A new perspective on the Sullivan dictionary

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joint work with Liam Stuart*

*who I also thank for help with the slides

Poincaré ball model of hyperbolic space

We model $(d+1)$-dimensional hyperbolic space with the ball

$$\mathbb{D}^{d+1} = \{z \in \mathbb{R}^{d+1} \ : \ |z| < 1\}$$

equipped with the hyperbolic metric $d_{\mathbb{H}}$ defined by

$$dt = \frac{2|dz|}{1-|z|^2}.$$

This is referred to as the Poincaré ball model. Denote the 'boundary at infinity' of $\mathbb{D}^{d+1}$ by

$$S^d = \{z \in \mathbb{R}^{d+1} \ : \ |z| = 1\}.$$

Isometries and Kleinian Groups

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Kleinian groups act 'properly discontinuously' on $\mathbb{D}^{d+1}$, but this may fail on (parts of) the boundary.

Limit Sets

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Let $\Gamma \leq \mathsf{Con}^+(d)$ *be a Kleinian group. Then the* **limit set** *of* Γ*, denoted as* $L(\Gamma)$*, is*

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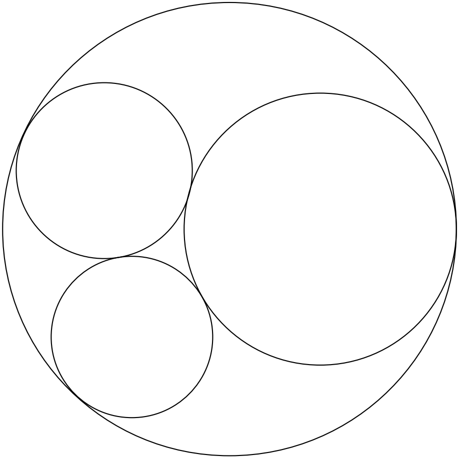
Limit sets capture the complexity of the action of the Kleinian group on the boundary of hyperbolic space. It is instructive to demonstrate that limit sets are closed, Γ-invariant, independent of choice of base point, and (provided they contain at least 3 points) are perfect. When the limit set is perfect, the group is called non-elementary.

Example



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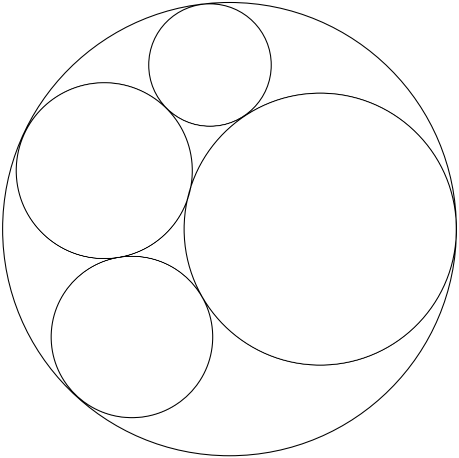

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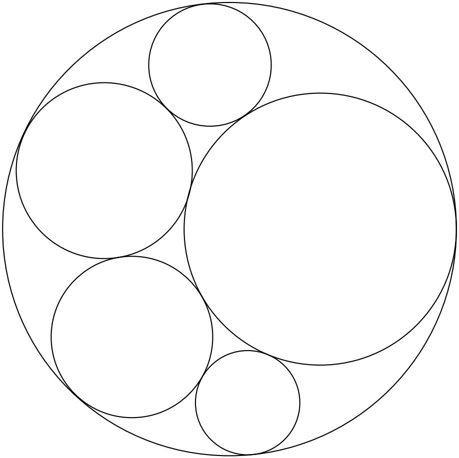

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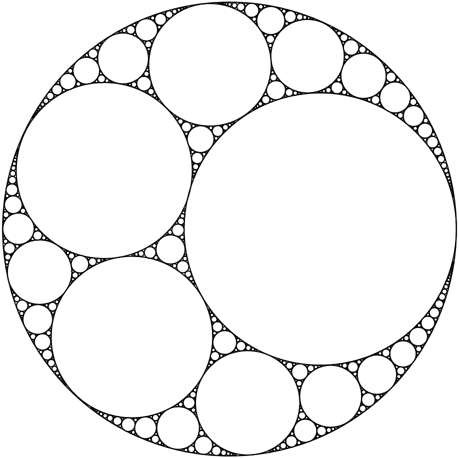

Figure: Apollonian gasket

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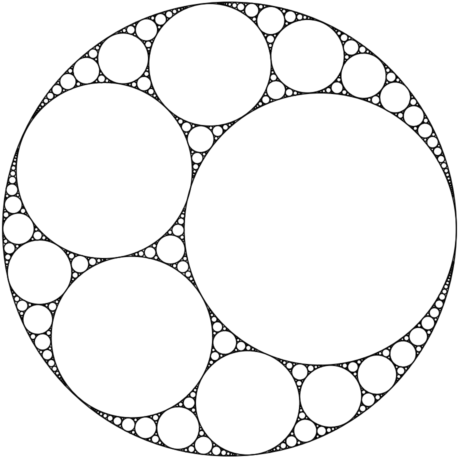

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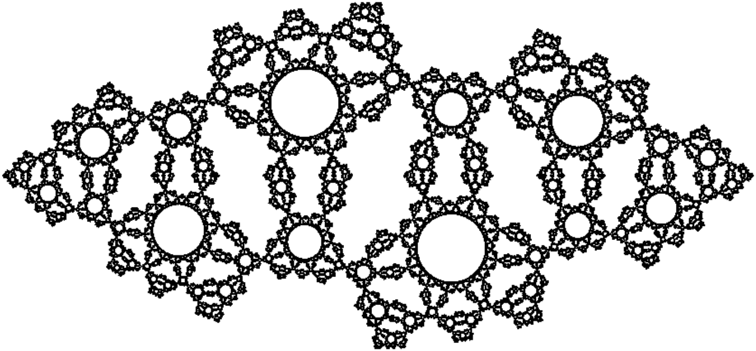
(Liam is good with tikz)

Another example



Geometric finiteness and Poincaré exponent

We restrict our attention to non-elementary geometrically finite Kleinian groups.

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Definition

The **Poincaré exponent** *of a Kleinian group* Γ *is given by*

$$\delta = \inf\left\{ s > 0 \mid \sum_{g \in \Gamma} e^{-s d_{\mathbb{H}}(0, g(0))} < \infty \right\}$$

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Theorem (Patterson '76, Sullivan '84)

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Theorem (Stratmann-Urbański '96, Bishop-Jones '97)

For a geometrically finite Kleinian group Γ,

$$\dim_{\mathrm{B}} L(\Gamma) = \dim_{\mathrm{P}} L(\Gamma) = \delta$$

Assouad dimensions of sets and measures

Definition

The **Assouad dimension** *of a set* $F \subset \mathbb{R}^d$ *is given by*

$$\dim_{\mathrm{A}} F = \inf\Big\{ s \geq 0 \mid \exists\, C > 0 \textit{ such that for all } x \in F,\ 0 < r < R,$$
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- $\dim_{\mathrm{A}} F = \dim^* F = \sup\{\dim_{\mathrm{H}} E : E \text{ is a microset of } F\}$

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Let μ be a locally finite Borel measure on $\mathbb{R}^d$.

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$$\dim_{\mathrm{A}}\mu = \inf\Big\{ s \geq 0 \mid \exists\, C > 0 \textit{ such that for all } x \in \textit{supp}(\mu),$$

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Proposition

For bounded $F \subset \mathbb{R}^d$,

$$\dim_{\mathrm{H}} F \leq \dim_{\mathrm{P}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}}^{\theta} F \leq \dim_{\mathrm{A}} F.$$

For locally finite Borel μ,

$$\dim_{\mathrm{H}} \mu \leq \dim_{\mathrm{A}}^{\theta} \mu \leq \dim_{\mathrm{A}} \mu.$$

Proposition

For closed $F \subset \mathbb{R}^d$,

$$\dim_{\mathrm{H}} F = \sup\{\dim_{\mathrm{H}} \mu \mid \mathrm{supp}(\mu) \subseteq F\}.$$

and

$$\dim_{\mathrm{A}} F = \inf\{\dim_{\mathrm{A}} \mu \mid \mathrm{supp}(\mu) = F\}.$$

Patterson-Sullivan measure

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Patterson-Sullivan measure, denoted by μ_{PS}, is a particularly good example. It is δ-conformal, Γ-ergodic and

$$\dim_{\mathsf{H}} \mu_{\mathsf{PS}} = \delta.$$

Rank of a parabolic point

Let P be the countable set of parabolic fixed points of a geometrically finite Kleinian group Γ, let $p \in P$ and consider $\text{Stab}(p) \leq \Gamma$.

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(Note that $\mathsf{Stab}(p)$ cannot contain any loxodromic elements, as this would violate discreteness.)
Let $k(p)$ be the maximal rank of a free abelian subgroup of $\mathsf{Stab}(p)$, i.e. the maximal integer n such that there exist $f_1, \dots, f_n \in \mathsf{Stab}(p)$ such that

$$\langle f_1, \dots, f_n \rangle \cong \mathbb{Z}^n.$$

We write

$$k_{\mathsf{max}} = \max\{k(p) \mid p \in P\}$$
$$k_{\mathsf{min}} = \min\{k(p) \mid p \in P\}.$$

Assouad dimension and Kleinian groups

Theorem (F '19, TAMS)

For a geometrically finite Kleinian group Γ*,*

$$\dim_{\mathrm{A}} L(\Gamma) = \max\{\delta, k_{\max}\}$$

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Punchline: the Assouad dimensions of the limit set and Patterson-Sullivan measure are *not necessarily* given by the Poincaré exponent.

Assoud dimension

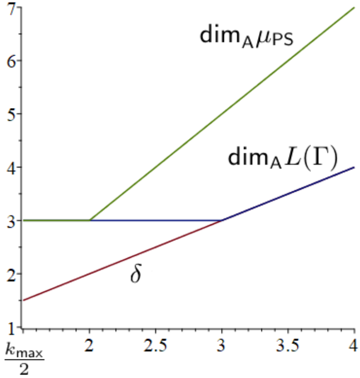

Figure: Plots of $\dim_A L(\Gamma)$ and $\dim_A \mu_{PS}$ as functions of δ with δ plotted for reference. Here $d = 4$, $k_{\max} = 3$, $k_{\min} = 1$.

Proof sketch for $L(\Gamma)$

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This is easy:

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The upper bound is rather harder and uses ideas from Diophantine approximation due to Stratmann-Velani, the Patterson-Sullivan measure, and "localised" analogues of covering arguments of Stratmann-Urbański.

The Assouad spectrum and Kleinian groups

Theorem (F+Stuart '20)

For a geometrically finite Kleinian group Γ *with* $\delta < k_{\mathsf{max}}$,

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(i) If $\delta < k_{\mathsf{min}}$*, then* $\dim_{\mathsf{A}}^{\theta} \mu_{\mathsf{PS}} = \dim_{\mathsf{A}}^{\theta} L(\Gamma)$*,*

(ii) if $k_{\mathsf{min}} \leq \delta < \frac{k_{\mathsf{min}}+k_{\mathsf{max}}}{2}$*, then*

$$\dim_{\mathsf{A}}^{\theta} \mu_{\mathsf{PS}} = 2\delta - k_{\mathsf{min}} + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\mathsf{max}} - (2\delta - k_{\mathsf{min}}))$$

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Punchline: the Assouad dimensions of the limit set and Patterson-Sullivan measure can be connected to the Poincaré exponent via the Assouad spectrum.

Assoud spectrum

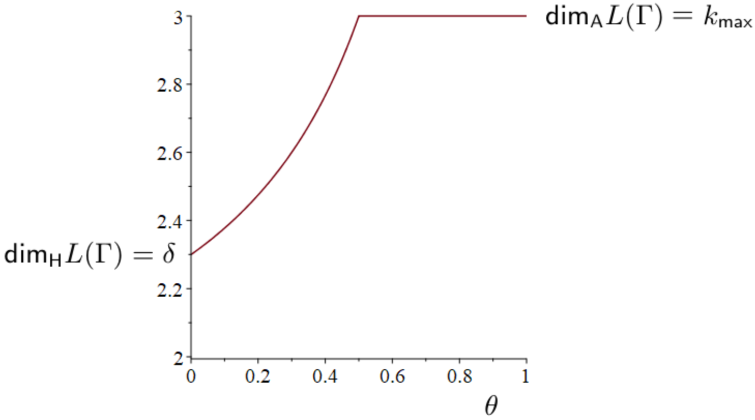

Figure: A plot of $\dim_{\mathrm{A}}^{\theta} L(\Gamma)$ for $\theta \in (0, 1)$.

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$$(T^p)'(\omega) = e^{2\pi i q} \text{ for some } q \in \mathbb{Q}.$$

We say that T is parabolic if $J(T)$ contains no critical points, but does contain at least one parabolic point. We write $\mathbf{\Omega}$ for the finite set of parabolic points. We may assume further that $T(\omega) = \omega$ and $T'(\omega) = 1$ for all $\omega \in \mathbf{\Omega}$.

Petal number

Let $\omega \in \mathbf{\Omega}$. On a sufficiently small neighbourhood of ω, there exists a unique holomorphic inverse branch T_ω^{-1} of T such that $T_\omega^{-1}(\omega) = \omega$ and

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We call $p(\omega)$ the **petal number** of ω, and write

$$p_{\min} = \min\{p(\omega) \mid \omega \in \boldsymbol{\Omega}\}$$
$$p_{\max} = \max\{p(\omega) \mid \omega \in \boldsymbol{\Omega}\}.$$

Example

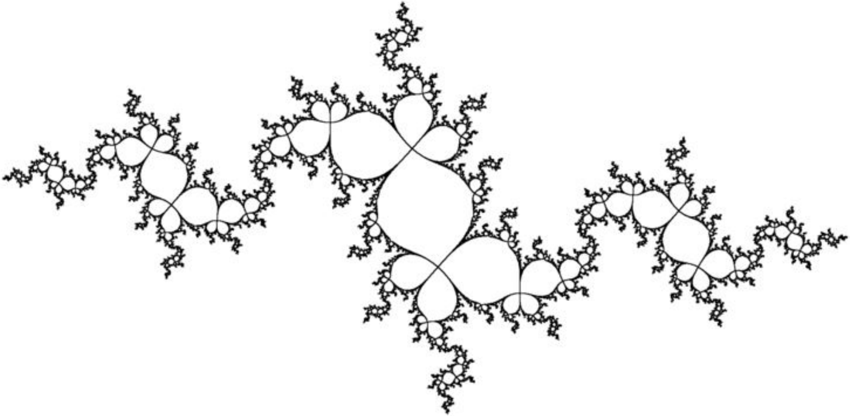

Figure: Parabolic Julia set

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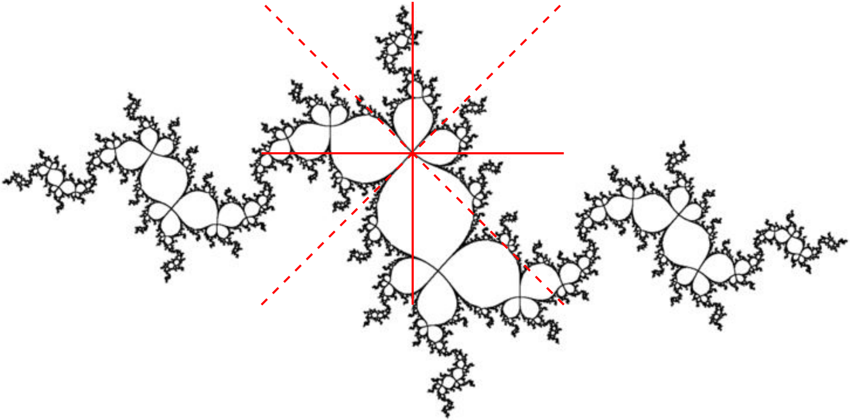

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h has many equivalent definitions, such as the smallest zero of the 'pressure function' $P(T, -t \log |T'|)$.

Dimension results

Theorem (Patterson '76, Sullivan '84, Stratmann-Urbański '96)

Let $\Gamma < \mathsf{Con}(d)$ *be a geometrically finite Kleinian group. Then*

$$\mathsf{dim_H} L(\Gamma) = \mathsf{dim_P} L(\Gamma) = \mathsf{dim_B} L(\Gamma) = \mathsf{dim_H} \mu_{\mathsf{PS}} = \delta.$$

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Theorem (Denker-Urbański '92, McMullen '00)

Let T *be a parabolic rational map. Then*

$$\dim_{\mathrm{H}} J(T) = \dim_{\mathrm{P}} J(T) = \dim_{\mathrm{B}} J(T) = \dim_{\mathrm{H}} m = h.$$

Here m is an h-conformal T-ergodic measure, which parallels Patterson-Sullivan measure.

Assouad dimension and Julia sets

Theorem (F-Stuart '20)

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Recall, for a geometrically finite Kleinian group Γ,

$$\mathsf{dim}_\mathsf{A} L(\Gamma) = \mathsf{max}\{k_{\mathsf{max}}, \delta\}$$

$$\mathsf{dim}_\mathsf{A} \mu_{\mathsf{PS}} = \mathsf{max}\{k_{\mathsf{max}}, 2\delta - k_{\mathsf{min}}\}$$

The Assoud spectrum and Julia sets

Theorem (F-Stuart '20)

Let T be a parabolic rational map with $h < 1$, and let $\theta \in (0,1)$. Then

$$\dim_{\mathrm{A}}^{\theta} J(T) = \dim_{\mathrm{A}}^{\theta} m = h + \min\left\{1, \frac{\theta\, p_{\max}}{1-\theta}\right\}(1-h)$$

Recall, for $\Gamma < \mathrm{Con}(d)$ be a geometrically finite Kleinian group with $\delta < k_{\max}$, and let $\theta \in (0,1)$,

$$\dim_{\mathrm{A}}^{\theta} L(\Gamma) = \delta + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\max} - \delta).$$

(i) If $\delta < k_{\min}$, then $\dim_{\mathrm{A}}^{\theta} \mu_{\mathrm{PS}} = \dim_{\mathrm{A}}^{\theta} L(\Gamma)$.

(ii) If $k_{\min} \leq \delta < \frac{k_{\min}+k_{\max}}{2}$, then

$$\dim_{\mathrm{A}}^{\theta} \mu_{\mathrm{PS}} = 2\delta - k_{\min} + \min\left\{1, \frac{\theta}{1-\theta}\right\}(k_{\max} - (2\delta - k_{\min})).$$

New entries in the Sullivan dictionary

1) *Interpolation between dimensions*

In both settings, the Assouad spectrum always interpolates between the upper box and Assouad dimensions of the respective sets and measures.

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where F can be replaced by μ, $L(\Gamma)$, m or $J(T)$. It turns out this formula also holds for other classes of sets such as Bedford-McMullen carpets, but does not hold in general (e.g. elliptical polynomial spirals, Burrell-Falconer-F '20).

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1) *Assouad dimension*

One stark difference between the Kleinian and Julia settings is that it is possible to have $\Gamma < \mathrm{Con}(d)$ such that $\dim_{\mathrm{A}} L(\Gamma) = d$, i.e. Kleinian limit sets can have full Assouad dimension.

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1) *Assouad dimension*

One stark difference between the Kleinian and Julia settings is that it is possible to have $\Gamma < \mathrm{Con}(d)$ such that $\dim_{\mathrm{A}} L(\Gamma) = d$, i.e. Kleinian limit sets can have full Assouad dimension.

However, as $\dim_{\mathrm{A}} J(T) = \max\{1, h\}$, combined with the fact that $h < 2$ (Aaronson-Denker-Urbański '93), we have $\dim_{\mathrm{A}} J(T) < 2$, and so parabolic Julia sets can never have full Assouad dimension.

New non-entries in the Sullivan dictionary

2) *Relationships between dimensions*

Recall

$$\dim_{\mathrm{A}} L(\Gamma) = \max\{\delta, k_{\max}\}$$

$$\dim_{\mathrm{L}} L(\Gamma) = \min\{\delta, k_{\min}\}$$

$$\dim_{\mathrm{A}} J(T) = \max\{1, h\}$$

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In particular, when $k_{\min} < \delta < k_{\max}$, we have

$$\dim_{\mathsf{L}} L(\Gamma) < \dim_{\mathsf{H}} L(\Gamma) < \dim_{\mathsf{A}} L(\Gamma)$$

which is not possible in the Julia setting.

New non-entries in the Sullivan dictionary

3) *Phase transition*

Turning our attention to the Assouad spectra, we recall the 'phase transition'

$$\rho = \inf\{\theta \in (0,1) \mid \dim_A^\theta F = \dim_A F\}.$$

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In the Julia setting, the phase transition is equal to $1/(1+p_{\max})$, and so depends on the rational map.

The global measure formula

Theorem (Global Measure Formula, Stratmann-Velani '95)

Let Γ *be a geometrically finite Kleinian group and* $\{H_p\}_{p \in P}$ *be a standard set of horoballs. Then for* $z \in L(\Gamma)$ *and* $T > 0$*,*

$$\mu_{\mathsf{PS}}(B(z, e^{-T})) \approx e^{-T\delta} e^{\rho(z,T)(k(z,T)-\delta)}$$

where

$$\rho(z, T) = k(z, T) = 0 \text{ if } z_T \notin H_p \text{ for all } p \in P$$

and

$$\rho(z, T) = \inf\{d_{\mathbb{H}}(z_T, y) \mid y \notin H_p\}$$
$$k(z, T) = k(p)$$

if $z_T \in H_p$ *for some* $p \in P$*.*

The global measure formula

Note that if $L(\Gamma)$ does not contain any parabolic points, then setting $R = e^{-T}$, we have

$$\mu_{\mathsf{PS}}(B(z,R)) \approx R^{\delta} \text{ for all } z \in L(\Gamma).$$

One can then easily show that

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One can then easily show that

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Therefore, the interesting case is when $L(\Gamma)$ contains parabolic points.

Global measure formulae

Theorem (Stratmann-Urbański '00)

Let $\xi \in J(T)$, $0 < r < |J(T)|$. *Then we have*

$$m(B(\xi, r)) \approx r^h \phi(\xi, r).$$

The values of ϕ *are determined as follows:*

i) Suppose $\xi \in J_r(T)$ has associated optimal sequence $(n_j(\xi))_{j \in \mathbb{N}}$ and hyperbolic zooms $(r_j(\xi))_{j \in \mathbb{N}}$ and r is such that $r_{j+1}(\xi) \le r < r_j(\xi)$ for some $j \in \mathbb{N}$ and $T^k(\xi) \in U_\omega$ for all $n_j(\xi) < k < n_{j+1}(\xi)$ and for some $\omega \in \mathbf{\Omega}$. Then

$$\phi(\xi, r) \approx \begin{cases} \left(\frac{r}{r_j(\xi)}\right)^{(h-1)p(\omega)} & r > r_j(\xi)\left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{\frac{1}{1+p(\omega)}} \\ \left(\frac{r_{j+1}(\xi)}{r}\right)^{h-1} & r \le r_j(\xi)\left(\frac{r_{j+1}(\xi)}{r_j(\xi)}\right)^{\frac{1}{1+p(\omega)}} \end{cases}.$$

Global measure formulae

Theorem (Stratmann-Urbański '00)

ii) Suppose $\xi \in J_p(T)$ has associated terminating optimal sequence $(n_j(\xi))_{j=1,\dots,l}$ and hyperbolic zooms $(r_j(\xi))_{j=1,\dots,l}$. Suppose $T^{n_l(\xi)}(\xi) = \omega$ for some $\omega \in \mathbf{\Omega}$. If $r > r_l(\xi)$, the values of ϕ are determined as in the radial case, and if $r \leq r_l(\xi)$, then

$$\phi(\xi, r) \approx \left(\frac{r}{r_l(\xi)} \right)^{(h-1)p(\omega)}.$$

Proof sketch for μ_{PS}

Upper bound: $\dim_{\mathsf{A}}^{\theta} \mu_{\mathsf{PS}} \leq \delta + \min\{1, \frac{\theta}{1-\theta}\}(k_{\max} - \delta)$ when $\delta < k_{\min}$.

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Suppose $z \in L(\Gamma)$, $T > 0$, $\theta \in (0, \frac{1}{2})$, and assume z_T and $z_{T\theta}$ lie in the same horoball H_p.

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By the global measure formula:

$$\frac{\mu_{\mathsf{PS}}(B(z, e^{-T\theta}))}{\mu_{\mathsf{PS}}(B(z, e^{-T}))} \approx \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta} e^{(\rho(z,T\theta)-\rho(z,T))(k(p)-\delta)}$$
$$\leq \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta} e^{T\theta(k_{\max}-\delta)}$$
$$= \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta + \frac{\theta}{1-\theta}(k_{\max}-\delta)}$$

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Lower bound: $\dim_{\mathsf{A}}^{\theta} \mu_{\mathsf{PS}} \geq \delta + \min\{1, \frac{\theta}{1-\theta}\}(k_{\max} - \delta)$ when $\delta < k_{\min}$.

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We can show that for sufficiently large T, we can choose $z \in L(\Gamma)$ such that

$$\rho(z,T) = k(z,T) = 0$$
$$k(z,T\theta) = k_{\max}$$
$$\rho(z,T\theta) \geq T\theta - C$$

for some constant C.

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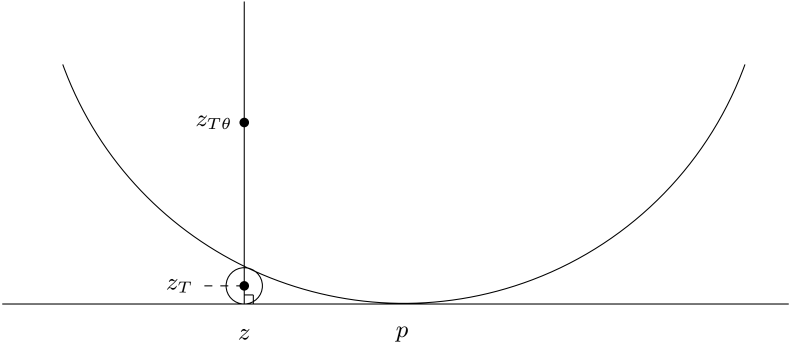
for some constant C. By the global measure formula:

$$\begin{aligned}
\frac{\mu_{\mathsf{PS}}(B(z,e^{-T\theta}))}{\mu_{\mathsf{PS}}(B(z,e^{-T}))} &\approx \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta} e^{(\rho(z,T\theta)-\rho(z,T))(k(z,T\theta)-\delta)} \\
&\gtrsim \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta} e^{T\theta(k_{\max}-\delta)} \\
&= \left(\frac{e^{-T\theta}}{e^{-T}}\right)^{\delta+\frac{\theta}{1-\theta}(k_{\max}-\delta)}
\end{aligned}$$

Proof sketch for μ_{PS}



Thank you for listening!

Figure: 'Circle Limit III' by M.C. Escher