On the analyticity of Falconer's subadditive pressure function

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2011

Singular values and subadditive pressure

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For $s \in [0, n)$ the *singular value function* $\phi^s : \mathcal{I}^* \to (0, \infty)$ is defined by

$$\phi^s(\mathbf{i}) \, = \, \alpha_1(\mathbf{i}) \, \alpha_2(\mathbf{i}) \, \cdots \, \alpha_m(\mathbf{i}) \, \alpha_{m+1}(\mathbf{i})^{s-m}$$

where $m \in \{0, \ldots, n-1\}$ is the unique non-negative integer satisfying $m \leqslant s < m+1$.

Singular values and subadditive pressure

The *subadditive pressure* $P : [0, n) \to \mathbb{R}$ is defined by

$$P(s) \; = \; \lim_{k \to \infty} \frac{1}{k} \log \sum_{\mathbf{i} \in \mathcal{I}^k} \phi^s(\mathbf{i}).$$

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It is convenient to extend the domain of P to $[0, \infty)$ and so we let

$$P(s) \;=\; \log \sum_{i\in\mathcal{I}} \det(A_i)^{s/n}$$

for $s \geqslant n$.

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(4) semi-differentiable everywhere

(5) differentiable at all but at most countably many points

Self-affine sets

To each of the matrices in our collection, associate a translation vector $t_i \in \mathbb{R}^n$. The collection of maps $\{A_i + t_i\}_{i \in \mathcal{I}}$ is an iterated function system and it is well-known that there is a unique non-empty compact set $F \subset \mathbb{R}^n$ satisfying

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$$F = \bigcup_{i \in \mathcal{I}} (A_i + t_i)(F).$$

Since each of the defining maps is *affine*, the attractor is called a *self-affine set*.

Falconer's Theorem

Theorem (Falconer '88, Solomyak '98)

Let s be the unique real number satisfying $P(s) = 0$. Then for all sets of translations $\{t_i\}_{i \in \mathcal{I}}$, we have

$$\dim_H F \;\leqslant\; \overline{\dim}_B F \;\leqslant\; \min\left\{n, s\right\}.$$

Assume in addition that the matrices all have Lipschitz constants strictly less than $1/2$. Then, for $\mathcal{L}^{|\mathcal{I}|n}$-almost all $(t_1, \dots, t_m) \in \mathbb{R}^{|\mathcal{I}|n}$, we have

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Kenneth's initial proof from 1988 required that the Lipschitz constants be strictly less than $1/3$ but this was relaxed to $1/2$ by Solomyak who also noted that $1/2$ is the optimal constant.

1989

Falconer's Theorem

Figure : Three self-affine sets with the same linear part but different translations. Falconer's theorem implies that they all have the same Hausdorff dimension, unless of course we have been very unlucky and chosen some 'exceptional parameters'.

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(6) Nonlinear analogues. (Falconer '94, Manning-Simon '07, Barańy '09)

Pressure in the conformal setting

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Let $\{S_i\}_{i \in \mathcal{I}}$ be a collection of conformal contractions on some open domain in $\mathbb{R}^n$. This time the unique non-empty compact set satisfying

$$F = \bigcup_{i \in \mathcal{I}} S_i(F)$$

is called a self-conformal set. This time define the pressure $P : [0, \infty) \to \mathbb{R}$ by

$$P(s) \ = \ \lim_{k \to \infty} \frac{1}{k} \log \sum_{\mathbf{i} \in \mathcal{I}^k} \mathrm{Lip}^+(S_{\mathbf{i}})^s.$$

Pressure in the conformal setting

Theorem (Bowen '75, Ruelle '78)

The pressure is real analytic on $(0, \infty)$ *and, writing* s *for the unique real number satisfying* $P(s) = 0$*, we have*

$$\dim_H F = \overline{\dim}_B F \leqslant \min\{n, s\}.$$

Assume in addition that the open set condition is satisfied. Then

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It is easy to see that the pressure will typically have phase transitions at the natural numbers $1, 2, \dots, n$, due to the change in definition of the singular value function.

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...examples to follow later.

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Theorem (F '13)

If the matrices $\{A_i\}_{i \in \mathcal{I}}$ *are simultaneously triangularisable, then the pressure is piecewise real analytic.*

2012

Reduction to diagonal matrices case

Theorem (Falconer-Miao '07)

If the matrices $\{A_i\}_{i \in \mathcal{I}}$ *are simultaneously triangularisable, then the pressure is the same as if one took the same collection of matrices but set all non-diagonal entries equal to zero.*

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Corollary

We only need to consider diagonal matrices!

A simple form for the pressure

Let $c_1(i), c_2(i), \ldots, c_n(i)$ be the diagonal entries of A_i.

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Let S_n be the symmetric group on $\{1, \ldots, n\}$.

For each $\sigma \in S_n$ and $s \in [0, n)$ define the *σ-ordered singular value function* $\phi_\sigma^s : \mathcal{I}^* \to (0, \infty)$ by

$$\phi_\sigma^s(\mathbf{i}) = c_{\sigma(1)}(\mathbf{i})\, c_{\sigma(2)}(\mathbf{i}) \cdots c_{\sigma(m)}(\mathbf{i})\, c_{\sigma(m+1)}(\mathbf{i})^{s-m}$$

where $m \in \{0, \ldots, n-1\}$ is the unique non-negative integer satisfying $m \leqslant s < m + 1$.

A simple form for the pressure

The key advantage of these ordered singular value functions is that they are multiplicative in $\mathbf{i}$ instead of only submultiplicative, i.e.

$$\phi^s_\sigma(\mathbf{i}\,\mathbf{j}) = \phi^s_\sigma(\mathbf{i})\,\phi^s_\sigma(\mathbf{j})$$

for all $\mathbf{i}$, $\mathbf{j} \in \mathcal{I}^*$ and $\sigma \in S_n$.

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This allows us to define the associated pressure by means of a closed form expression, without taking a limit. More precisely, we define the *σ-ordered pressure* $P_\sigma : [0, n) \to \mathbb{R}$ by

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...which is real analytic on $(0, \infty) \setminus \{1, 2, \ldots, n\}$ for each σ.

A simple form for the pressure

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Manning-Simon '07 proved this in the 2-dimensional case.

Falconer-Miao '07 gave a similar result in n-dimensions, but they took the maximum over a larger family of functions.

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The pressure is piecewise real analytic on $(0, \infty)$.

This follows from the 'principle of permanence'.

However, it would be more useful to have explicit bounds on the number of phase transitions (in terms of n and $|\mathcal{I}|$?)

Bounding the number of phase transitions

Let $\sigma, \tau \in S_n$.

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Assuming the ordered pressures P_σ and P_τ are not equal on the entire interval $(m, m+1)$ we will show how to bound the number of times their graphs can intersect.

Let

$$E(s) \;=\; \sum_{i \in \mathcal{I}} \phi_\sigma^s(\mathbf{i}) - \sum_{i \in \mathcal{I}} \phi_\tau^s(\mathbf{i})$$

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for some $a_i \in \mathbb{R}$ and $b_i > 0$ and $N \leqslant 2|\mathcal{I}|$.

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for some $a_i \in \mathbb{R}$ and $b_i > 0$ and $N \leqslant 2|\mathcal{I}|$.

Thus $E(s)$ is a (generalised) Dirichlet polynomial and therefore can have at most $2|\mathcal{I}| - 1$ zeros on $\mathbb{R}$.

Bounding the number of phase transitions

Let $m \in \{0, \ldots, n-1\}$. In the interval $(m, m+1)$, there are at most

$$\begin{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \cdot \begin{pmatrix} n-m \\ 1 \end{pmatrix} \\ \\ 2 \end{pmatrix}$$

distinct pairs of ordered pressures, each of which can cross at most $2|\mathcal{I}| - 1$ times.

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Summing over m this yields an upper bound of

$$\big(2|\mathcal{I}| - 1\big) \sum_{m=0}^{n-1} \binom{(n-m)\binom{n}{m}}{2}$$

on the total number of crossings of the ordered pressures.

Bounding the number of phase transitions

Since every phase transition for the pressure corresponds to a crossing of some pair of ordered pressures, and remembering the n potential integer phase transitions, we can bound the total number of phase transitions for the subadditive pressure by

$$n + (2|\mathcal{I}| - 1) \sum_{m=0}^{n-1} \binom{(n-m)\binom{n}{m}}{2} \sim \frac{n\sqrt{n}\,4^n}{8\sqrt{\pi}}$$

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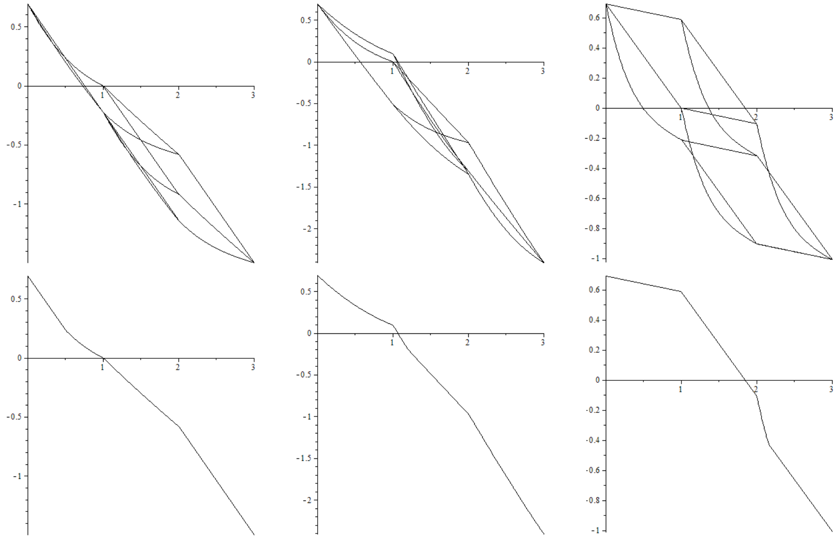
$$n \; + \; (2|\mathcal{I}| - 1) \sum_{m=0}^{n-1} \left(\begin{array}{c} (n-m) \left(\begin{array}{c} n \\ m \end{array} \right) \\ \\ 2 \end{array} \right) \sim \frac{n\sqrt{n}\,4^n}{8\sqrt{\pi}}$$

and so the pressure is piecewise real analytic on $(0, \infty)$.

Examples of non-integer phase transitions



Future work

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Is the subadditive pressure always piecewise real analytic or at least piecewise differentiable?

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Is there any interesting geometric or dynamical significance of the ordered pressures in regions where they are strictly less than the subadditive pressure?

Happy birthday Kenneth!

Main references

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