The Assouad dimension of self-similar sets with overlaps

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Such sets are called self-similar sets.

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It turns out that if F is the attractor of $\{S_i\}_{i \in \mathcal{I}}$, then

$$\dim_H F \; \leqslant \; \dim_{\mathrm{sim}} \{S_i\}_{i \in \mathcal{I}}.$$

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This separation makes the geometry of the IFS and its attractor F much simpler. Indeed, if the OSC is satisfied, then

$$\dim_H F \ = \ \dim_{\text{sim}} \{S_i\}_{i \in \mathcal{I}}.$$

A big question

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$$\dim^*_{\mathrm{sim}} F \quad := \quad \inf \Big\{ \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} : \{S_i\}_{i\in\mathcal{I}} \text{is an IFS of similarities generating } F \Big\}$$

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Folklore?

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Peres-Solomyak 1998

Is it true that

$$\dim_H F < \dim_{\mathrm{sim}} \{S_i\}_{i \in \mathcal{I}} \Rightarrow \mathrm{Semi}\langle S_i : i \in \mathcal{I} \rangle \text{ is not free?}$$

Hochman's Theorem

We say that $\{S_i\}_{i \in \mathcal{I}}$ has *super-exponential concentration of cylinders* if $-\log \Delta_k / k \to \infty$, where

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If $\{S_i\}_{i\in\mathcal{I}}$ does not have super-exponential concentration of cylinders, then

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If the defining parameters for $\{S_i\}_{i \in \mathcal{I}}$ are algebraic, then

$$\dim_H F \;<\; \dim_{\mathrm{sim}} \{S_i\}_{i \in \mathcal{I}} \quad \Rightarrow \quad \mathrm{Semi}\langle S_i : i \in \mathcal{I} \rangle \quad \text{is not free.}$$

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- In fact the initial motivation was to prove the following theorem: a metric space can be quasisymmetrically embedded into some Euclidean space if and only if it has finite Assouad dimension.

  Robinson: *Dimensions, Embeddings, and Attractors*
  Heinonen: *Lectures on Analysis on Metric Spaces*.

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- The Assouad dimension gives 'coarse but local' information about a set, unlike the Hausdorff dimension which gives 'fine but global' information.

The Assoud dimension

The *Assouad dimension* of a non-empty subset F of X is defined by

$$\dim_{\mathrm{A}} F \; = \; \inf \Bigg\{ \; \alpha \; : \; \text{there exists constants } C, \, \rho > 0 \text{ such that,} \\ \text{for all } 0 < r < R \leqslant \rho, \text{ we have} \\ \sup_{x \in F} N_r\big(B(x,R) \cap F\big) \; \leqslant \; C\left(\frac{R}{r}\right)^{\alpha} \; \Bigg\}.$$

Relationships between dimensions

For $F \subseteq X$, we have

$$\begin{array}{ccccccc} & & \dim_{\mathrm{P}} F & & & & \\ & \rotatebox{45}{$\leqslant$} & & \rotatebox{-45}{$\leqslant$} & & & \\ \dim_{\mathrm{H}} F & & & & \overline{\dim}_{\mathrm{B}} F & \leqslant & \dim_{\mathrm{A}} F. \\ & \rotatebox{-45}{$\leqslant$} & & \rotatebox{45}{$\leqslant$} & & & \\ & & \underline{\dim}_{\mathrm{B}} F & & & & \end{array}$$

Ahlfors regular sets

Recall that a compact set F is called Ahlfors regular if for all $x \in F$

$$\mathcal{H}^{\dim_H F}\big(B(x,r)\big) \ \asymp \ r^{\dim_H F}$$

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For Ahflors regular sets F all the standard notions of dimension coincide, in particular,

$$\dim_H F \ = \ \dim_A F.$$

Basic properties

Property	$\dim_{\mathrm{H}}$	$\dim_{\mathrm{P}}$	$\underline{\dim}_{\mathrm{B}}$	$\overline{\dim}_{\mathrm{B}}$	$\dim_{\mathrm{A}}$
Monotone	✓	✓	✓	✓	✓
Finitely stable	✓	✓	×	✓	✓
Countably stable	✓	✓	×	×	×
Lipschitz stable	✓	✓	✓	✓	×
Bi-Lipschitz stable	✓	✓	✓	✓	✓
Stable under taking closures	×	×	✓	✓	✓
Open set property	✓	✓	✓	✓	✓
Measurable	✓	×	✓	✓	✓

Back to self-similar sets

It is well-known that any self-similar set (regardless of overlaps) satisfies:

$$\dim_{\mathrm{H}} F \;=\; \underline{\dim}_{\mathrm{B}} F \;=\; \overline{\dim}_{\mathrm{B}} F \;=\; \dim_{\mathrm{P}} F \;\leqslant\; \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}}$$

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Olsen ('12) asked if the Assouad dimension of a self-similar set with overlaps can ever exceed the upper box dimension.

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Answer:

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This example is from my PhD thesis. Similar examples, not exactly in the context of Assouad dimension, were known before by András Máthé and Tuomas Orponen.

Proof

Let $\alpha, \beta, \gamma \in (0,1)$ be such that $(\log \beta)/(\log \alpha) \notin \mathbb{Q}$ and define similarity maps S_1, S_2, S_3 on $[0,1]$ as follows

$$S_1(x) = \alpha x, \qquad S_2(x) = \beta x \qquad \text{and} \qquad S_3(x) = \gamma x + (1-\gamma).$$

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Let F be the self-similar attractor of $\{S_1, S_2, S_3\}$. We will now prove that $\dim_A F = 1$ and, in particular, the Assouad dimension is independent of α, β, γ provided they are chosen with the above property.

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Proposition

Let $X \subset \mathbb{R}$ be compact and let F be a compact subset of X. Let T_k be a sequence of similarity maps defined on $\mathbb{R}$ and suppose that $T_k(F) \cap X \to_{d_{\mathcal{H}}} \hat{F}$ for some non-empty compact set $\hat{F}$. Then $\dim_A \hat{F} \leqslant \dim_A F$. The set $\hat{F}$ is called a weak tangent to F.

Proof

We will now show that $[0, 1]$ is a weak tangent to F in the above sense. Let $X = [0, 1]$ and assume without loss of generality that $\alpha < \beta$. For each $k \in \mathbb{N}$ let T_k be defined by

$$T_k(x) = \beta^{-k}x.$$

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Since

$$E_k \ := \ \left\{\alpha^m \beta^n : m \in \mathbb{N}, n \in \{-k, \dots, \infty\}\right\} \cap [0,1] \ \subset \ T_k(F) \cap [0,1]$$

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and Dirichlet's Theorem gives that there exists infinitely many n such that

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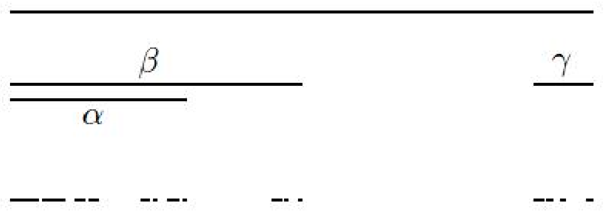
with n arbitrarily large. We can thus make $m \log \alpha + n \log \beta$ arbitrarily small and this gives the result.

Proof

If we choose α, β, γ such that $\dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} < 1$, then

$$\dim_{\mathrm{H}} F \;\leqslant\; \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} \;<\; 1 \;=\; \dim_{\mathrm{A}} F.$$



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Theorem (Zerner 1996)

If F is the self-similar attractor of an IFS satisfying the WSP, then

$$\dim_H F = \dim_{sim}^* F.$$

Different separation conditions

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$$\text{freeness} \;\Leftrightarrow\; Id \notin \mathcal{E}$$

Our main result

Theorem (F., Henderson, Olsen, Robinson 2014)

Let F be a self-similar subset of $\mathbb{R}$.

- *If the WSP is satisfied, then F is Ahlfors regular and so*

$$\dim_A F \;=\; \dim_H F \;=\; \dim^*_{sim} F.$$

- *If the WSP is not satisfied, then*

$$\dim_A F \;=\; 1.$$

Higher dimensions?

What could the analogous result be for self-similar sets in $\mathbb{R}^d$?

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Conjecture (??)

Let F be a self-similar subset of $\mathbb{R}^d$, not contained in any hyperplane.

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- *If the WSP is satisfied, then F is Ahlfors regular and so*

$$\dim_A F = \dim_H F = \dim^*_{sim} F. \quad \textit{TRUE!}$$

- *If the WSP is not satisfied, then*

$$\dim_A F = d. \quad \textit{FALSE!}$$

What we can prove

Theorem (F., Henderson, Olsen, Robinson 2014)

Let F be a self-similar subset of $\mathbb{R}^d$, not contained in any hyperplane.

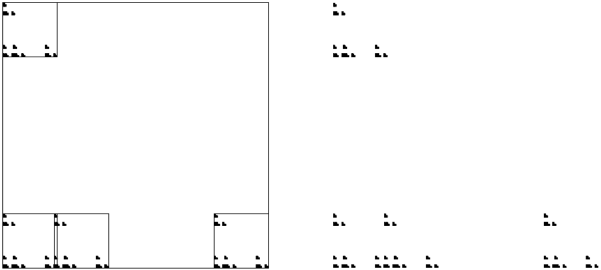
- *If the WSP is satisfied, then F is Ahlfors regular and so*

$$\dim_A F \ = \ \dim_H F \ = \ \dim^*_{sim} F.$$

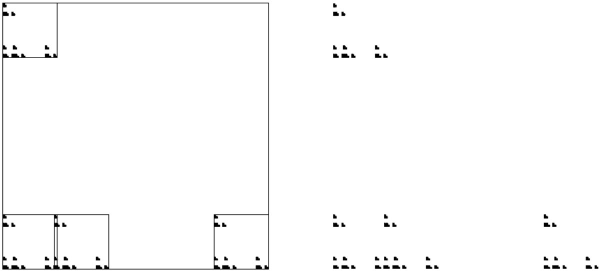
- *If the WSP is not satisfied, then*

$$\dim_A F \ \geqslant \ 1.$$

An example

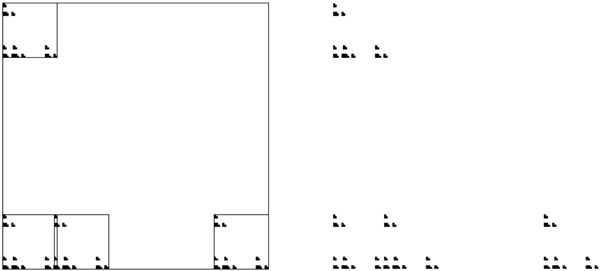


An example



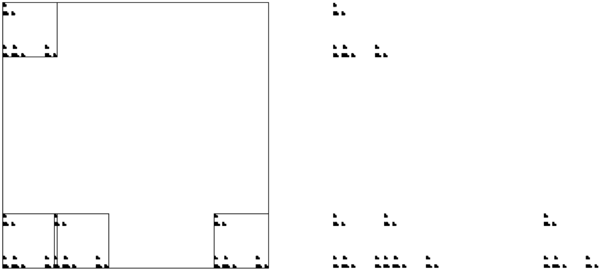
$\dim_H F$

An example



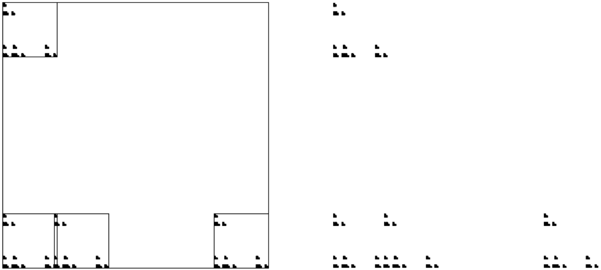
$$\dim_H F \quad \leqslant \quad \dim_{\text{sim}} \{S_i\}_{i \in \mathcal{I}}$$

An example



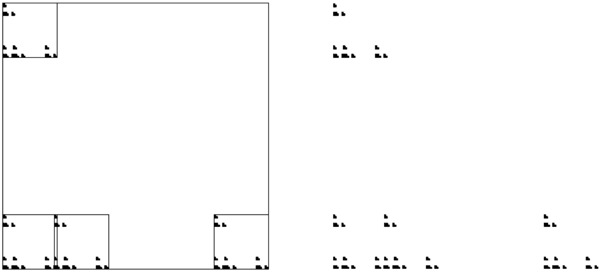
$$\dim_H F \quad \leqslant \quad \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} = \frac{\log 4}{\log 5}$$

An example



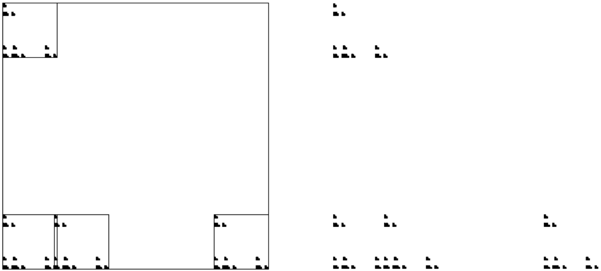
$$\dim_H F \quad \leqslant \quad \dim_{\text{sim}}\{S_i\}_{i\in\mathcal{I}} = \frac{\log 4}{\log 5} < 1$$

An example



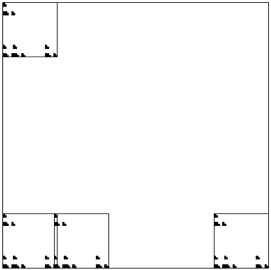
$$\dim_H F \quad \leqslant \quad \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} = \frac{\log 4}{\log 5} < 1 = \dim_A \pi_1 F$$

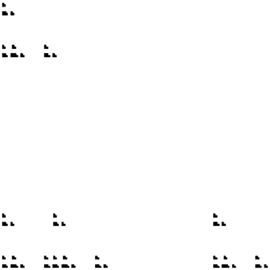
An example



$$\begin{aligned} \dim_H F \quad &\leqslant \quad \dim_{\mathrm{sim}}\{S_i\}_{i\in\mathcal{I}} = \frac{\log 4}{\log 5} < 1 = \dim_A \pi_1 F \\ &\leqslant \quad \dim_A F \end{aligned}$$

An example

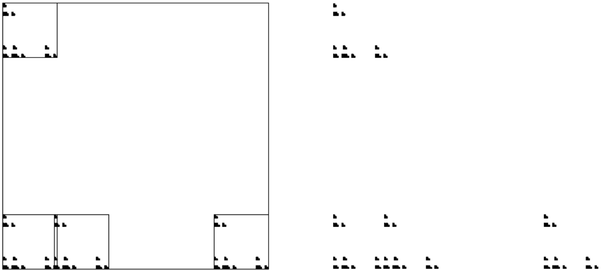




$$\dim_H F \leqslant \dim_{\text{sim}}\{S_i\}_{i\in\mathcal{I}} = \frac{\log 4}{\log 5} < 1 = \dim_A \pi_1 F$$

$$\leqslant \dim_A F \leqslant \dim_A \pi_1 F \times \pi_2 F$$

An example



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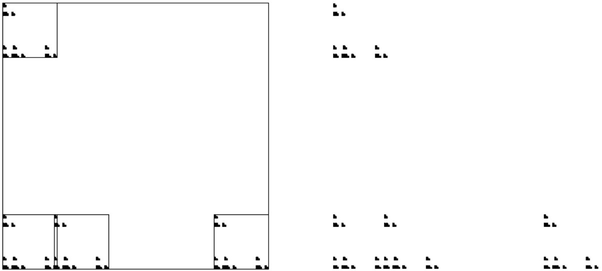
$$\leqslant \dim_A F \leqslant \dim_A \pi_1 F \times \pi_2 F \leqslant \dim_A \pi_1 F + \dim_A \pi_2 F$$

An example



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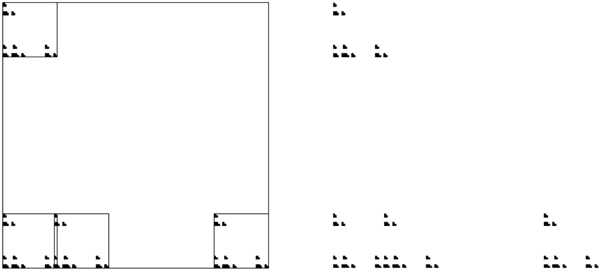
An example



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$$= 1 + \frac{\log 2}{\log 5} < 2$$

Future work

- Sufficient conditions for

$$\dim_A F \geqslant k$$

for self-similar F in $\mathbb{R}^d$ and $k \leqslant d$?

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• Sufficient conditions for

$$\dim_A F \;\geqslant\; k$$

for self-similar F in $\mathbb{R}^d$ and $k \leqslant d$?

• Exploring 'maximal v something' dichotomies for Assouad dimension in other settings.

Thank you!

Main references

- J. M. Fraser. Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc., to appear*, (2013), arXiv: 1301.2934.
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