On the L^q-spectrum of planar self-affine measures

Jonathan M. Fraser

The University of Warwick

L^q-spectra

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ with support denoted by $F = \operatorname{supp} \mu$.

L^q-spectra

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ with support denoted by $F = \operatorname{supp}\mu$.

The *L^q-spectrum* of μ is defined by

$$\tau_\mu(q) = \lim_{\delta \to 0} \frac{\log \int_F \mu\big(B(x,\delta)\big)^{q-1} \, d\mu(x)}{-\log \delta}$$

with $q \in \mathbb{R}$.

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ with support denoted by $F = \operatorname{supp} \mu$.

The L^q*-spectrum* of μ is defined by

$$\tau_\mu(q) = \lim_{\delta \to 0} \frac{\log \int_F \mu\big(B(x,\delta)\big)^{q-1} \, d\mu(x)}{-\log \delta}$$

with $q \in \mathbb{R}$.

This spectrum gives a quantitative analysis of the global fluctuations of μ.

L^q-spectra

Let μ be a compactly supported Borel probability measure on $\mathbb{R}^n$ with support denoted by $F = \operatorname{supp} \mu$.

The L^q*-spectrum* of μ is defined by

$$\tau_\mu(q) = \lim_{\delta \to 0} \frac{\log \int_F \mu\big(B(x,\delta)\big)^{q-1} \, d\mu(x)}{-\log \delta}$$

with $q \in \mathbb{R}$.

This spectrum gives a quantitative analysis of the global fluctuations of μ.

The motivation to study this spectrum has roots in information theory.

Basic properties of the L^q-spectrum

τ_μ is . . .

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$ (easy to see)

Basic properties of the L^q-spectrum

τ_μ is . . .

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$ (easy to see)

- Lipschitz on $[\lambda, \infty)$ for any $\lambda > 0$

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$ (easy to see)

- Lipschitz on $[\lambda, \infty)$ for any $\lambda > 0$ (consequence of convexity)

Basic properties of the L^q-spectrum

τ_μ is . . .

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$ (easy to see)

- Lipschitz on $[\lambda, \infty)$ for any $\lambda > 0$ (consequence of convexity)

- differentiable on $(0, \infty)$ at all but at most countably many points and semi-differentiable everywhere

Basic properties of the L^q-spectrum

τ_μ is ...

- convex, and thus continuous, on $(0, \infty)$ (apply Hölder's inequality)

- decreasing on $[0, \infty)$ (easy to see)

- equal to 0 at $q = 1$ (easy to see)

- Lipschitz on $[\lambda, \infty)$ for any $\lambda > 0$ (consequence of convexity)

- differentiable on $(0, \infty)$ at all but at most countably many points and semi-differentiable everywhere (consequence of convexity)

Why do people interested in fractals study the L^q-spectrum?

Why do people interested in fractals study the L^q-spectrum?

Relationship to the dimension theory of F and μ ...

Why do people interested in fractals study the L^q-spectrum?

Relationship to the dimension theory of F and μ ...

$$\dim_{\mathrm{B}} F = \tau(0)$$

Why do people interested in fractals study the L^q-spectrum?

Relationship to the dimension theory of F and μ ...

$$\dim_{\mathrm{B}} F = \tau(0)$$

If τ is differentiable at $q = 1$, then

$$\dim_{\mathrm{H}} \mu = \dim_{\mathrm{P}} \mu = \dim_e \mu = -\tau'(1)$$

Why do people interested in fractals study the L^q-spectrum?

Relationship to the dimension theory of F and μ . . .

$$\dim_{\mathrm{B}} F = \tau(0)$$

If τ is differentiable at $q = 1$, then

$$\dim_{\mathrm{H}} \mu = \dim_{\mathrm{P}} \mu = \dim_{e} \mu = -\tau'(1) \qquad \text{(Ngai '97)}$$

Why do people interested in fractals study the L^q-spectrum?

Relationship to the dimension theory of F and μ ...

$$\dim_{\mathrm{B}} F = \tau(0)$$

If τ is differentiable at $q = 1$, then

$$\dim_{\mathrm{H}} \mu = \dim_{\mathrm{P}} \mu = \dim_e \mu = -\tau'(1) \qquad \text{(Ngai '97)}$$

and therefore

$$-\tau'(1) \leqslant \dim_{\mathrm{H}} F \leqslant \dim_{\mathrm{P}} F \leqslant \tau(0).$$

Why do people interested in fractals study the L^q-spectrum?

Relationship to the multifractal structure of μ . . .

Why do people interested in fractals study the L^q-spectrum?

Relationship to the multifractal structure of μ . . .

Let

$$\Delta_\alpha = \{x \in F : \dim_{\mathrm{loc}} \mu(x) = \alpha\}$$

for $\alpha \geqslant 0$, where $\dim_{\mathrm{loc}} \mu(x)$ is the local dimension of μ at x, if it exists.

Why do people interested in fractals study the L^q-spectrum?

Relationship to the multifractal structure of μ ...

Let

$$\Delta_\alpha = \{x \in F : \dim_{\text{loc}} \mu(x) = \alpha\}$$

for $\alpha \geqslant 0$, where $\dim_{\text{loc}} \mu(x)$ is the local dimension of μ at x, if it exists.

The Hausdorff and packing multifractal spectra are defined by

$$f_{\mathrm{H},\mu}(\alpha) = \dim_{\mathrm{H}} \Delta_\alpha$$

and

$$f_{\mathrm{P},\mu}(\alpha) = \dim_{\mathrm{P}} \Delta_\alpha$$

for $\alpha \geqslant 0$.

Why do people interested in fractals study the L^q-spectrum?

Relationship to the multifractal structure of μ ...

Let

$$\Delta_\alpha = \{x \in F : \dim_{\mathrm{loc}} \mu(x) = \alpha\}$$

for $\alpha \geqslant 0$, where $\dim_{\mathrm{loc}} \mu(x)$ is the local dimension of μ at x, if it exists.

The Hausdorff and packing multifractal spectra are defined by

$$f_{\mathrm{H},\mu}(\alpha) = \dim_{\mathrm{H}} \Delta_\alpha$$

and

$$f_{\mathrm{P},\mu}(\alpha) = \dim_{\mathrm{P}} \Delta_\alpha$$

for $\alpha \geqslant 0$.

For all $\alpha \geqslant 0$, we have

$$f_{\mathrm{H},\mu}(\alpha) \leqslant f_{\mathrm{P},\mu}(\alpha) \leqslant \tau_\mu^*(\alpha)$$

Why do people interested in fractals study the L^q-spectrum?

Relationship to the multifractal structure of μ ...

Let

$$\Delta_\alpha = \{x \in F : \dim_{\text{loc}} \mu(x) = \alpha\}$$

for $\alpha \geqslant 0$, where $\dim_{\text{loc}} \mu(x)$ is the local dimension of μ at x, if it exists.

The Hausdorff and packing multifractal spectra are defined by

$$f_{\mathrm{H},\mu}(\alpha) = \dim_{\mathrm{H}} \Delta_\alpha$$

and

$$f_{\mathrm{P},\mu}(\alpha) = \dim_{\mathrm{P}} \Delta_\alpha$$

for $\alpha \geqslant 0$.

For all $\alpha \geqslant 0$, we have

$$f_{\mathrm{H},\mu}(\alpha) \leqslant f_{\mathrm{P},\mu}(\alpha) \leqslant \tau_\mu^*(\alpha) \qquad \text{(for example, Olsen '95, '98)}$$

An example

Let μ be a self-similar measure

$$\mu = \sum_i p_i \, \mu \circ S_i^{-1},$$

satisfying the strong separation condition, with defining probabilities $p_i \in (0,1)$ and similarity mappings S_i with contraction ratios equal to c_i

An example

Let μ be a self-similar measure

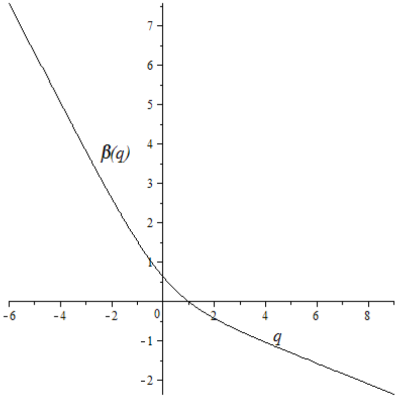
$$\mu = \sum_i p_i \, \mu \circ S_i^{-1},$$

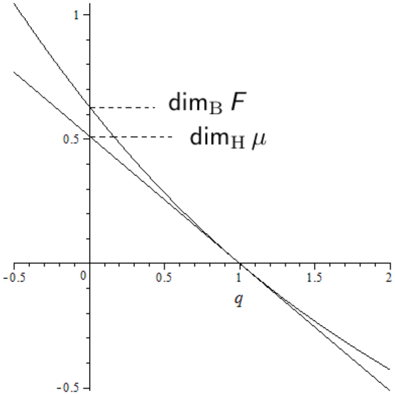
satisfying the strong separation condition, with defining probabilities $p_i \in (0,1)$ and similarity mappings S_i with contraction ratios equal to c_i

The L^q-spectra of μ is given by the unique function $\beta : \mathbb{R} \to \mathbb{R}$ defined by

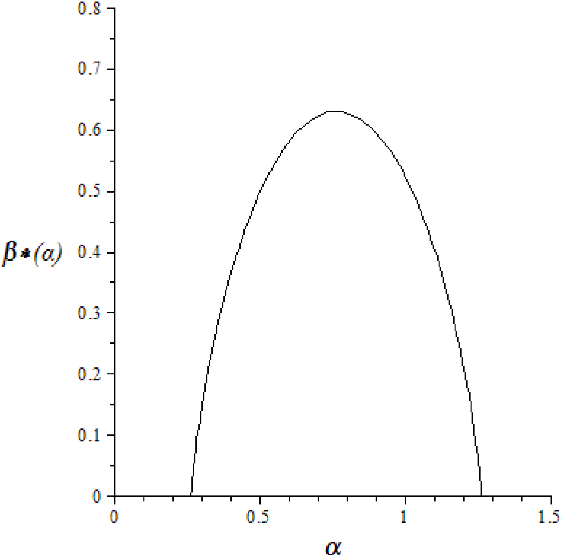
$$\sum_i p_i^q \, c_i^{\beta(q)} = 1$$

An example





An example



What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

- asymptotic formula

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

- asymptotic formula

Feng-Wang '05: much more general class of self-affine carpet

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

- asymptotic formula

Feng-Wang '05: much more general class of self-affine carpet

- supremum of an expression over a simplex of probability vectors

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

- asymptotic formula

Feng-Wang '05: much more general class of self-affine carpet

- supremum of an expression over a simplex of probability vectors

Falconer '10: generic result for almost self-affine measures in the range $(1, \infty]$

What about self-affine measures?

Olsen '98: Sierpiński sponges for all $q \in \mathbb{R}$

- needed a strong separation condition
- explicit formula

Falconer '99: generic result in the range $(1, 2]$

- asymptotic formula

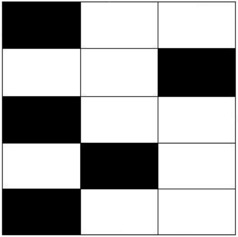
Feng-Wang '05: much more general class of self-affine carpet

- supremum of an expression over a simplex of probability vectors

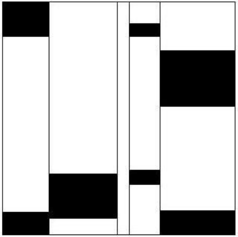
Falconer '10: generic result for almost self-affine measures in the range $(1, \infty]$

- asymptotic formula

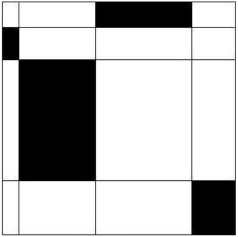
Self-affine carpets



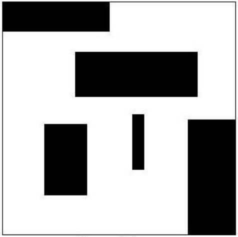
Bedford-McMullen



Gatzouras-Lalley



Barański



Feng-Wang

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet

$$\tau_\mu(q) = \max\{\theta_A, \theta_B\}$$

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet

$$\tau_\mu(q) = \max\{\theta_A, \theta_B\}$$

where

$$\theta_A = \sup_{\mathbf{t} \in \Gamma_A} \frac{\mathbf{t} \cdot \Big(\log \mathbf{t} + \tau_{\pi_2(\mu)}(q)(\log \mathbf{d} - \log \mathbf{c}) - q \log \mathbf{p} \Big)}{\mathbf{t} \cdot \log \mathbf{d}}$$

and

$$\theta_B = \sup_{\mathbf{t} \in \Gamma_B} \frac{\mathbf{t} \cdot \Big(\log \mathbf{t} + \tau_{\pi_1(\mu)}(q)(\log \mathbf{c} - \log \mathbf{d}) - q \log \mathbf{p} \Big)}{\mathbf{t} \cdot \log \mathbf{c}}$$

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet where $c_i \geqslant d_i$ for all i,

$$\sum_i p_i^q c_i^{\tau_{\pi_1(\mu)}(q)} d_i^{\tau_\mu(q) - \tau_{\pi_1(\mu)}(q)} = 1.$$

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet where $c_i \geqslant d_i$ for all i,

$$\sum_i p_i^q c_i^{\tau_{\pi_1(\mu)}(q)} d_i^{\tau_\mu(q) - \tau_{\pi_1(\mu)}(q)} = 1.$$

In this case we have a **closed form expression** for the spectrum.

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet where $c_i \geqslant d_i$ for all i,

$$\sum_i p_i^q c_i^{\tau_{\pi_1(\mu)}(q)} d_i^{\tau_\mu(q) - \tau_{\pi_1(\mu)}(q)} = 1.$$

In this case we have a **closed form expression** for the spectrum.

This allows precise analysis of differentiability properties and gives applications concerning the Hausdorff dimension of μ and F.

Feng and Wang's result

Theorem (Feng-Wang '05)

Let $q \geqslant 0$. For a self-affine measure on a Feng-Wang carpet where $c_i \geqslant d_i$ for all i,

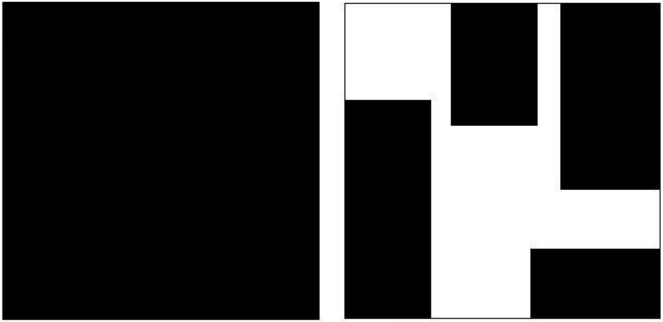
$$\sum_i p_i^q c_i^{\tau_{\pi_1(\mu)}(q)} d_i^{\tau_\mu(q)-\tau_{\pi_1(\mu)}(q)} = 1.$$

In this case we have a **closed form expression** for the spectrum.

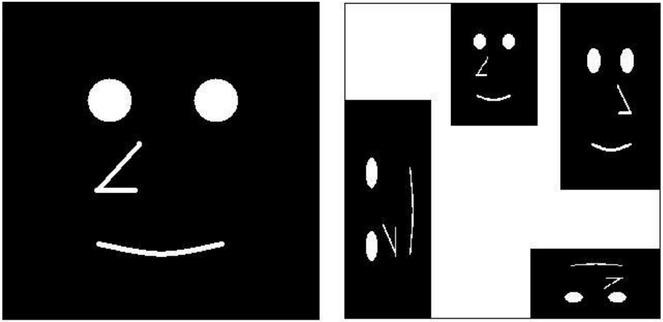
This allows precise analysis of differentiability properties and gives applications concerning the Hausdorff dimension of μ and F.

In fact, the spectrum is differentiable for all $q \in (0, \infty)$.

Our class of measures



Our class of measures



Our class of measures

Our class of measures

For $q \geqslant 0$, let

$$\tau_1(q) = \tau_{\pi_1(\mu)}(q)$$

and

$$\tau_2(q) = \tau_{\pi_2(\mu)}(q).$$

Our class of measures

For $q \geqslant 0$, let

$$\tau_1(q) = \tau_{\pi_1(\mu)}(q)$$

and

$$\tau_2(q) = \tau_{\pi_2(\mu)}(q).$$

If there are orientation reversing maps in the IFS, then these are a pair of graph-directed self-similar measures,

Our class of measures

For $q \geqslant 0$, let

$$\tau_1(q) = \tau_{\pi_1(\mu)}(q)$$

and

$$\tau_2(q) = \tau_{\pi_2(\mu)}(q).$$

If there are orientation reversing maps in the IFS, then these are a pair of graph-directed self-similar measures, and they may have complicated overlaps.

Our class of measures

For $q \geqslant 0$, let
$$\tau_1(q) = \tau_{\pi_1(\mu)}(q)$$
and
$$\tau_2(q) = \tau_{\pi_2(\mu)}(q).$$

If there are orientation reversing maps in the IFS, then these are a pair of graph-directed self-similar measures, and they may have complicated overlaps.

Theorem (F '13, Peres-Solomyak '00)

The L^q-spectrum exists for $q \geqslant 0$ for any graph-directed self-similar measure, regardless of separation conditions.

q-modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $\tau_{\mathbf{i}}(q)$ be the L^q-spectrum of the projection of μ onto the longest side of the rectangle $S_{\mathbf{i}}\big([0,1]^2\big)$,

q-modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $\tau_{\mathbf{i}}(q)$ be the L^q-spectrum of the projection of μ onto the longest side of the rectangle $S_{\mathbf{i}}\big([0,1]^2\big)$, and note that this is always equal to either $\tau_1(q)$ or $\tau_2(q)$.

q-modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $\tau_{\mathbf{i}}(q)$ be the L^q-spectrum of the projection of μ onto the longest side of the rectangle $S_{\mathbf{i}}\big([0,1]^2\big)$, and note that this is always equal to either $\tau_1(q)$ or $\tau_2(q)$.

For $s \in \mathbb{R}$ and $q \geqslant 0$ and $\mathbf{i} \in \mathcal{I}^*$, we define the *$q$-modified singular value function*, $\psi^{s,q}$, by

$$\psi^{s,q}(\mathbf{i}) \;=\; p_{\mathbf{i}}^q\,\alpha_1(\mathbf{i})^{\tau_{\mathbf{i}}(q)}\;\alpha_2(\mathbf{i})^{s-\tau_{\mathbf{i}}(q)},$$

q-modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $\tau_{\mathbf{i}}(q)$ be the L^q-spectrum of the projection of μ onto the longest side of the rectangle $S_{\mathbf{i}}\big([0,1]^2\big)$, and note that this is always equal to either $\tau_1(q)$ or $\tau_2(q)$.

For $s \in \mathbb{R}$ and $q \geqslant 0$ and $\mathbf{i} \in \mathcal{I}^*$, we define the *$q$-modified singular value function*, $\psi^{s,q}$, by

$$\psi^{s,q}(\mathbf{i}) \;=\; p_{\mathbf{i}}^q\, \alpha_1(\mathbf{i})^{\tau_{\mathbf{i}}(q)}\; \alpha_2(\mathbf{i})^{s-\tau_{\mathbf{i}}(q)},$$

and for $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we define a number $\Psi_k^{s,q}$ by

$$\Psi_k^{s,q} = \sum_{\mathbf{i} \in \mathcal{I}^k} \psi^{s,q}(\mathbf{i})$$

q-modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $\tau_{\mathbf{i}}(q)$ be the L^q-spectrum of the projection of μ onto the longest side of the rectangle $S_{\mathbf{i}}([0,1]^2)$, and note that this is always equal to either $\tau_1(q)$ or $\tau_2(q)$.

For $s \in \mathbb{R}$ and $q \geqslant 0$ and $\mathbf{i} \in \mathcal{I}^*$, we define the *q-modified singular value function*, $\psi^{s,q}$, by

$$\psi^{s,q}(\mathbf{i}) \;=\; p_{\mathbf{i}}^q \, \alpha_1(\mathbf{i})^{\tau_{\mathbf{i}}(q)} \; \alpha_2(\mathbf{i})^{s-\tau_{\mathbf{i}}(q)},$$

and for $s \in \mathbb{R}$ and $k \in \mathbb{N}$, we define a number $\Psi_k^{s,q}$ by

$$\Psi_k^{s,q} = \sum_{\mathbf{i} \in \mathcal{I}^k} \psi^{s,q}(\mathbf{i})$$

We may define a function $P : \mathbb{R} \times [0,\infty) \to [0,\infty)$ by:

$$P(s,q) = \lim_{k\to\infty} (\Psi_k^{s,q})^{1/k}$$

A formula for the L^q-spectrum

For each $q \geqslant 0$, there is a unique value $s \in \mathbb{R}$ for which $P(s, q) = 1$

A formula for the L^q-spectrum

For each $q \geqslant 0$, there is a unique value $s \in \mathbb{R}$ for which $P(s, q) = 1$ and hence we may define a function $\gamma : [0, \infty) \to \mathbb{R}$ by

$$P(\gamma(q), q) = 1$$

A formula for the L^q-spectrum

For each $q \geqslant 0$, there is a unique value $s \in \mathbb{R}$ for which $P(s, q) = 1$ and hence we may define a function $\gamma : [0, \infty) \to \mathbb{R}$ by

$$P(\gamma(q), q) = 1$$

Unfortunately, the definition for $\gamma(q)$ is not a closed form expression.

A formula for the L^q-spectrum

For each $q \geqslant 0$, there is a unique value $s \in \mathbb{R}$ for which $P(s, q) = 1$ and hence we may define a function $\gamma : [0, \infty) \to \mathbb{R}$ by

$$P(\gamma(q), q) = 1$$

Unfortunately, the definition for $\gamma(q)$ is not a closed form expression.

However, $\gamma(q)$ can be numerically estimated by approximating it by functions γ_k defined by

$$\Psi_k^{\gamma_k(q),q} \;=\; \sum_{\mathbf{i} \in \mathcal{I}^k} p_{\mathbf{i}}^q \, \alpha_1(\mathbf{i})^{\tau_{\mathbf{i}}(q)} \; \alpha_2(\mathbf{i})^{\gamma_k(q) - \tau_{\mathbf{i}}(q)} \;=\; 1.$$

A formula for the L^q-spectrum

Lemma (Properties of γ)

(1) γ *is strictly decreasing on* $[0, \infty)$

(2) γ *is continuous on* $(0, \infty)$

A formula for the L^q-spectrum

Lemma (Properties of γ)

(1) γ *is strictly decreasing on* $[0, \infty)$

(2) γ *is continuous on* $(0, \infty)$

(3) γ *is the pointwise limit of* γ_k *as* $k \to \infty$

Lemma (Properties of γ)

(1) γ *is strictly decreasing on* $[0, \infty)$

(2) γ *is continuous on* $(0, \infty)$

(3) γ *is the pointwise limit of* γ_k *as* $k \to \infty$

(4) $\gamma(1) = 0$ *and* $\lim_{q\to\infty} \gamma(q) = -\infty$

(5) γ *is convex on* $(0, \infty)$

A formula for the L^q-spectrum

Theorem (F '13)

Let μ be in our class of measures. Then

(1) *For all $q \in [0, 1]$ we have*

$$\overline{\tau}_\mu(q) \leqslant \gamma(q).$$

(2) *For all $q \geqslant 1$ we have*

$$\gamma(q) \leqslant \underline{\tau}_\mu(q).$$

(3) *If μ satisfies the rectangular open set condition, then for all $q \geqslant 0$ we have*

$$\tau_\mu(q) = \gamma(q).$$

A closed form expression in the orientation preserving case

The main drawback of our formula is that it is not a closed form expression and this prevents us analysing differentiability of the spectrum.

A closed form expression in the orientation preserving case

The main drawback of our formula is that it is not a closed form expression and this prevents us analysing differentiability of the spectrum. However, thankfully we can use our result to get a closed form expression in the orientation preserving case (which includes the Feng-Wang class).

A closed form expression in the orientation preserving case

The main drawback of our formula is that it is not a closed form expression and this prevents us analysing differentiability of the spectrum. However, thankfully we can use our result to get a closed form expression in the orientation preserving case (which includes the Feng-Wang class).

Assume μ is 'orientation preserving', which means that the linear part of each map S_i in the defining IFS is of the form

$$\begin{pmatrix} \pm c_i & 0 \\ 0 & \pm d_i \end{pmatrix}$$

for constants $c_i, d_i \in (0,1)$, which are the singular values of S_i.

A closed form expression in the orientation preserving case

The main drawback of our formula is that it is not a closed form expression and this prevents us analysing differentiability of the spectrum. However, thankfully we can use our result to get a closed form expression in the orientation preserving case (which includes the Feng-Wang class).

Assume μ is 'orientation preserving', which means that the linear part of each map S_i in the defining IFS is of the form

$$\begin{pmatrix} \pm c_i & 0 \\ 0 & \pm d_i \end{pmatrix}$$

for constants $c_i, d_i \in (0,1)$, which are the singular values of S_i. Define $\gamma_A, \gamma_B : [0,\infty) \to \mathbb{R}$ by

$$\sum_{i \in \mathcal{I}} p_i^q \, c_i^{\tau_1(q)} \, d_i^{\gamma_A(q)-\tau_1(q)} = 1$$

and

$$\sum_{i \in \mathcal{I}} p_i^q \, d_i^{\tau_2(q)} \, c_i^{\gamma_B(q)-\tau_2(q)} = 1.$$

A closed form expression in the orientation preserving case

Since γ_A and γ_B are given by closed form expressions, it is easy to study their differentiability.

A closed form expression in the orientation preserving case

Since γ_A and γ_B are given by closed form expressions, it is easy to study their differentiability.

Lemma

If τ_1 is differentiable at $q > 0$, then γ_A is differentiable at q, with

$$\gamma_A'(q) = -\frac{\sum_{i \in \mathcal{I}} p_i^q \, c_i^{\tau_1(q)} \, d_i^{\gamma_A(q) - \tau_1(q)} \log \left(p_i c_i^{\tau_1'(q)} d_i^{-\tau_1'(q)} \right)}{\sum_{i \in \mathcal{I}} p_i^q \, c_i^{\tau_1(q)} \, d_i^{\gamma_A(q) - \tau_1(q)} \log d_i}$$

and if τ_2 is differentiable at $q > 0$, then γ_B is differentiable at q with a similar explicit formula.

A closed form expression in the orientation preserving case

Theorem (F '13)

(1) *If* $\max\{\gamma_A(q), \gamma_B(q)\} \leqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) = \max\{\gamma_A(q), \gamma_B(q)\}.$$

(2) *If* $\min\{\gamma_A(q), \gamma_B(q)\} \geqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) \leqslant \min\{\gamma_A(q), \gamma_B(q)\},$$

A closed form expression in the orientation preserving case

Theorem (F '13)

(1) *If* $\max\{\gamma_A(q), \gamma_B(q)\} \leqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) = \max\{\gamma_A(q), \gamma_B(q)\}.$$

(2) *If* $\min\{\gamma_A(q), \gamma_B(q)\} \geqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) \leqslant \min\{\gamma_A(q), \gamma_B(q)\},$$

with equality occurring if either of the following conditions are satisfied:

(2.1) $\sum_{i \in \mathcal{I}} p_i^q \, c_i^{\tau_1(q)} \, d_i^{\gamma_A(q) - \tau_1(q)} \, \log\left(c_i/d_i\right) \geqslant 0,$

(2.2) $\sum_{i \in \mathcal{I}} p_i^q \, d_i^{\tau_2(q)} \, c_i^{\gamma_B(q) - \tau_2(q)} \, \log\left(d_i/c_i\right) \geqslant 0.$

A closed form expression in the orientation preserving case

Theorem (F '13)

(1) *If* $\max\{\gamma_A(q), \gamma_B(q)\} \leqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) = \max\{\gamma_A(q), \gamma_B(q)\}.$$

(2) *If* $\min\{\gamma_A(q), \gamma_B(q)\} \geqslant \tau_1(q) + \tau_2(q)$, *then*

$$\gamma(q) \leqslant \min\{\gamma_A(q), \gamma_B(q)\},$$

with equality occurring if either of the following conditions are satisfied:

(2.1) $\sum_{i\in\mathcal{I}} p_i^q \, c_i^{\tau_1(q)} \, d_i^{\gamma_A(q)-\tau_1(q)} \, \log\left(c_i/d_i\right) \geqslant 0,$

(2.2) $\sum_{i\in\mathcal{I}} p_i^q \, d_i^{\tau_2(q)} \, c_i^{\gamma_B(q)-\tau_2(q)} \, \log\left(d_i/c_i\right) \geqslant 0.$

Moreover, if $c_i \geqslant d_i$ *for all* $i \in \mathcal{I}$, *then* $\gamma(q) = \gamma_A(q)$ *for all* $q \geqslant 0$, *and if* $d_i \geqslant c_i$ *for all* $i \in \mathcal{I}$, *then* $\gamma(q) = \gamma_B(q)$ *for all* $q \geqslant 0$, *without any additional assumptions.*

A closed form expression in the orientation preserving case

Theorem (F '13)

Let μ be of separated type and assume that τ_1 and τ_2 are differentiable at $q = 1$. Then γ is differentiable at $q = 1$ with

$$\gamma'(1) = \begin{cases} \min\{\gamma_A'(1), \gamma_B'(1)\} & \textit{if } \min\{\gamma_A'(1), \gamma_B'(1)\} \geqslant \tau_1'(1) + \tau_2'(1) \\ \max\{\gamma_A'(1), \gamma_B'(1)\} & \textit{if } \max\{\gamma_A'(1), \gamma_B'(1)\} \leqslant \tau_1'(1) + \tau_2'(1) \end{cases}$$

A closed form expression in the orientation preserving case

Theorem (F '13)

Let μ be of separated type and assume that τ_1 and τ_2 are differentiable at $q = 1$. Then γ is differentiable at $q = 1$ with

$$\gamma'(1) = \begin{cases} \min\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \min\{\gamma_A'(1), \gamma_B'(1)\} \geqslant \tau_1'(1) + \tau_2'(1) \\ \max\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \max\{\gamma_A'(1), \gamma_B'(1)\} \leqslant \tau_1'(1) + \tau_2'(1) \end{cases}$$

Corollary (F '13)

Let μ be of separated type and assume it satisfies the ROSC. Then

$$\dim_B F = \dim_P F = \max\{\gamma_A(0), \gamma_B(0)\}$$

A closed form expression in the orientation preserving case

Theorem (F '13)

Let μ be of separated type and assume that τ_1 and τ_2 are differentiable at $q = 1$. Then γ is differentiable at $q = 1$ with

$$\gamma'(1) = \begin{cases} \min\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \min\{\gamma_A'(1), \gamma_B'(1)\} \geqslant \tau_1'(1) + \tau_2'(1) \\ \max\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \max\{\gamma_A'(1), \gamma_B'(1)\} \leqslant \tau_1'(1) + \tau_2'(1) \end{cases}$$

Corollary (F '13)

Let μ be of separated type and assume it satisfies the ROSC. Then

$\dim_B F = \dim_P F = \max\{\gamma_A(0), \gamma_B(0)\}$ *(See also Barański '07 and F '12)*

A closed form expression in the orientation preserving case

Theorem (F '13)

Let μ be of separated type and assume that τ_1 and τ_2 are differentiable at $q = 1$. Then γ is differentiable at $q = 1$ with

$$\gamma'(1) = \begin{cases} \min\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \min\{\gamma_A'(1), \gamma_B'(1)\} \geqslant \tau_1'(1) + \tau_2'(1) \\ \max\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \max\{\gamma_A'(1), \gamma_B'(1)\} \leqslant \tau_1'(1) + \tau_2'(1) \end{cases}$$

Corollary (F '13)

Let μ be of separated type and assume it satisfies the ROSC. Then

$\dim_B F = \dim_P F = \max\{\gamma_A(0), \gamma_B(0)\}$ *(See also Barański '07 and F '12)*

If τ_1 and τ_2 are differentiable at $q = 1$, then

$$\dim_H \mu = \dim_P \mu = \dim_e \mu = -\gamma'(1)$$

A closed form expression in the orientation preserving case

Theorem (F '13)

Let μ be of separated type and assume that τ_1 and τ_2 are differentiable at $q = 1$. Then γ is differentiable at $q = 1$ with

$$\gamma'(1) = \begin{cases} \min\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \min\{\gamma_A'(1), \gamma_B'(1)\} \geqslant \tau_1'(1) + \tau_2'(1) \\ \max\{\gamma_A'(1), \gamma_B'(1)\} & \text{if } \max\{\gamma_A'(1), \gamma_B'(1)\} \leqslant \tau_1'(1) + \tau_2'(1) \end{cases}$$

Corollary (F '13)

Let μ be of separated type and assume it satisfies the ROSC. Then

$\dim_B F = \dim_P F = \max\{\gamma_A(0), \gamma_B(0)\}$ *(See also Barański '07 and F '12)*

If τ_1 and τ_2 are differentiable at $q = 1$, then

$$\dim_H \mu = \dim_P \mu = \dim_e \mu = -\gamma'(1)$$

which is equal to either $-\gamma_A'(1)$ or $-\gamma_B'(1)$.

A closed form expression in the orientation preserving case

If $c_i \geqslant d_i$ for all $i \in \mathcal{I}$,

A closed form expression in the orientation preserving case

If $c_i \geqslant d_i$ for all $i \in \mathcal{I}$, then

$$\dim_B F = \dim_P F = \gamma_A(0)$$

A closed form expression in the orientation preserving case

If $c_i \geqslant d_i$ for all $i \in \mathcal{I}$, then

$$\dim_B F = \dim_P F = \gamma_A(0)$$

and if τ_1 is differentiable at $q = 1$, then

$$\dim_H \mu = \dim_P \mu = \dim_e \mu = -\gamma_A'(1) = -\frac{\sum_{i \in \mathcal{I}} p_i \big(\log p_i + \tau_1'(1) \log(c_i/d_i) \big)}{\sum_{i \in \mathcal{I}} p_i \log d_i}.$$

A closed form expression in the orientation preserving case

If $c_i \geqslant d_i$ for all $i \in \mathcal{I}$, then

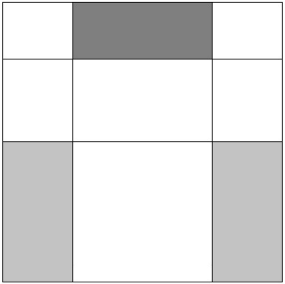
$$\dim_B F = \dim_P F = \gamma_A(0)$$

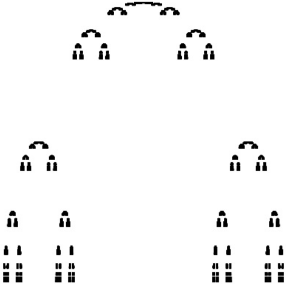
and if τ_1 is differentiable at $q = 1$, then

$$\dim_H \mu = \dim_P \mu = \dim_e \mu = -\gamma_A'(1) = -\frac{\sum_{i \in \mathcal{I}} p_i \big(\log p_i + \tau_1'(1) \log(c_i/d_i) \big)}{\sum_{i \in \mathcal{I}} p_i \log d_i}.$$

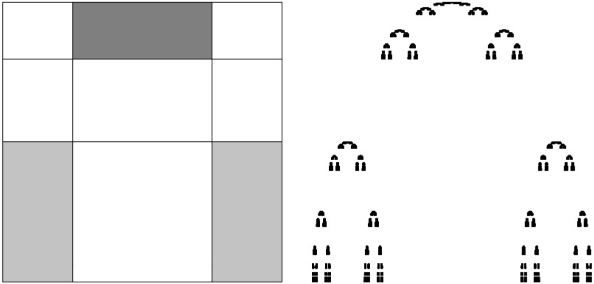
There is a similar formula if $c_i \leqslant d_i$ for all $i \in \mathcal{I}$.

An example





An example



The probability vector is $(3/5, 1/5, 1/5)$ and the unit square has been divided up into columns of widths $1/4, 1/2$ and $1/4$ and rows of heights $1/2, 3/10$ and $2/10$.

An example

We have a closed form expression for γ for all $q \in [0, \infty)$

An example

We have a closed form expression for γ for all $q \in [0, \infty)$

It turns out that γ has a phase transition at a point $q_0 \approx 0.237$, where it is not differentiable, but for all other values of $q \geqslant 0$ it is differentiable.

An example

We have a closed form expression for γ for all $q \in [0, \infty)$

It turns out that γ has a phase transition at a point $q_0 \approx 0.237$, where it is not differentiable, but for all other values of $q \geqslant 0$ it is differentiable.

$\gamma(q) = \gamma_B(q)$ for $q \in [0, q_0]$

$\gamma(q) = \gamma_A(q)$ for $q \in [q_0, \infty)$.

An example

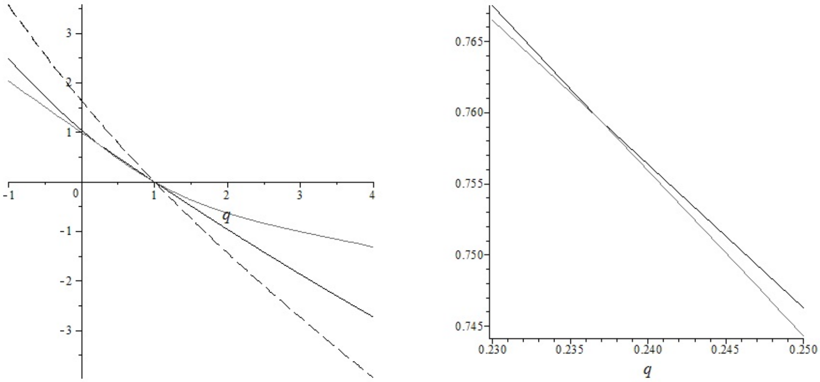


Figure : Left: The graph of γ (black), the graphs of the parts of γ_A and γ_B not equal to γ (grey), and the graph of $(\tau_1 + \tau_2)$ (dashed), which is included to indicate which of γ_A, γ_B is equal to γ, i.e., the one 'nearer' to $(\tau_1 + \tau_2)$.

An example

We also have closed form expressions for the dimensions.

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{P}} F = \gamma(0) = \gamma_B(0) = 1.046105401$$

and

$$\dim_{\mathrm{H}} \mu = \dim_{\mathsf{P}} \mu = \dim_{\mathsf{e}} \mu = -\gamma'(1) = -\gamma_A'(1) = 0.9792504246.$$

Further questions

Question

In the separated case, if $\min\{\gamma_A(q), \gamma_B(q)\} \geqslant \tau_1(q) + \tau_2(q)$ *and neither (2.1) nor (2.2) is satisfied, is it still true that*

$$\gamma(q) = \min\{\gamma_A(q), \gamma_B(q)\}?$$

Further questions

Question

In the separated case, if $\min\{\gamma_A(q), \gamma_B(q)\} \geqslant \tau_1(q) + \tau_2(q)$ *and neither (2.1) nor (2.2) is satisfied, is it still true that*

$$\gamma(q) = \min\{\gamma_A(q), \gamma_B(q)\}?$$

Even in the awkward situations where we do not have equality, our result still provides useful computational information as

$$\tau_1(q) + \tau_2(q) \ \leqslant \ \gamma_k(q) \ \leqslant \ \gamma(q) \ \leqslant \ \min\{\gamma_A(q), \gamma_B(q)\}$$

for all $k \in \mathbb{N}$.

Further questions

It would be interesting to consider negative values of q. The first question concerns the projections.

Further questions

It would be interesting to consider negative values of q. The first question concerns the projections.

Question

Do the L^q-spectra of (graph-directed) self-similar measures exist for all $q \in \mathbb{R}$?

Further questions

It would be interesting to consider negative values of q. The first question concerns the projections.

Question

Do the L^q-spectra of (graph-directed) self-similar measures exist for all $q \in \mathbb{R}$?

If the answer is 'yes', then we can at least define a moment scaling function as in the positive case.

Further questions

It would be interesting to consider negative values of q. The first question concerns the projections.

Question

Do the L^q-spectra of (graph-directed) self-similar measures exist for all $q \in \mathbb{R}$?

If the answer is 'yes', then we can at least define a moment scaling function as in the positive case.

However, precise calculations for negative q are very awkward.

Thank you!

Main references

- K. Barański. Hausdorff dimension of the limit sets of some planar geometric constructions, *Adv. Math.*, **210**, (2007), 215–245.
- D.-J. Feng and Y. Wang. A class of self-affine sets and self-affine measures, *J. Fourier Anal. Appl.*, **11**, (2005), 107–124.
- J. M. Fraser. On the packing dimension of box-like self-affine sets in the plane, *Nonlinearity*, **25**, (2012), 2075–2092.
- J. M. Fraser. On the L^q-spectrum of planar self-affine measures, *preprint*, (2013).
- L. Olsen. Self-affine multifractal Sierpiński sponges in $\mathbb{R}^d$, *Pacific J. Math*, **183**, (1998), 143–199.