The Kakeya problem

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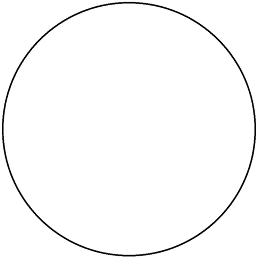

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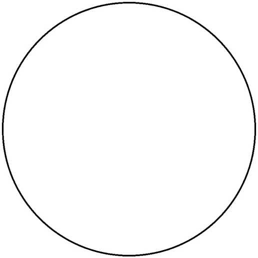

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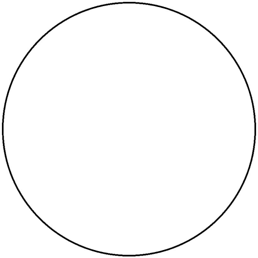

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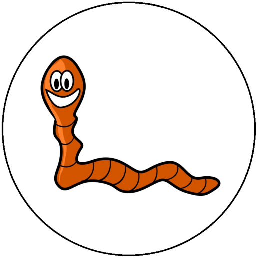

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But Besicovitch proved in 1919 that one can find examples with **arbitrarily small area!**

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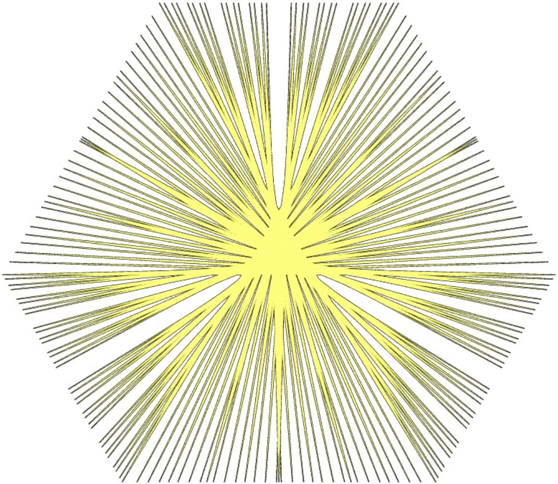

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What does this even mean?

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Classic notions of dimension and measure do not apply to fractals and so we have to invent new ones!

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assuming this limit exists! Otherwise we define upper and lower box dimension $\dim_{\mathrm{UB}} F$ and $\dim_{\mathrm{LB}} F$.

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This is often referred to as the **weak Kakeya problem**.

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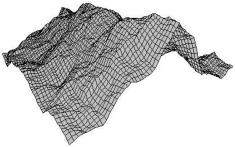
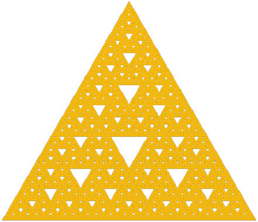
My co-authors

Fractals!!