Inhomogeneous iterated function systems

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Fractal Geometry and Stochastics V

Iterated function systems

Let X be a compact metric space. An **iterated function system (IFS)** on X is a finite collection $\{S_i\}_{i\in\mathcal{I}}$ of contracting self-maps on X. It is a fundamental result in fractal geometry that there exists a unique non-empty compact set F, called the **attractor**, which satisfies

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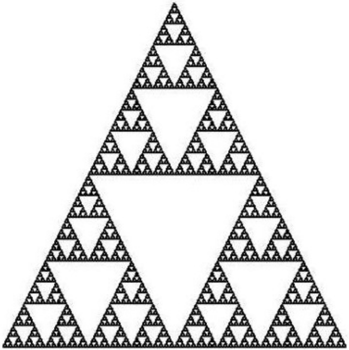
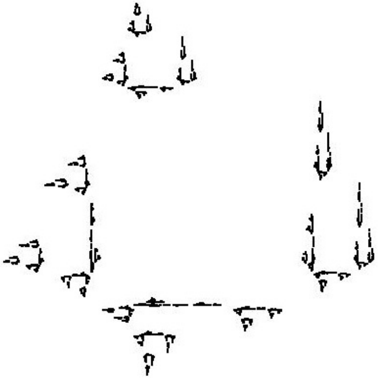
Common examples include: self-similar sets, self-affine sets, self-conformal sets, etc ...

Iterated function systems - examples





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From now on we will write homogeneous attractors as $F_\emptyset$, i.e. as inhomogeneous attractors with $C = \emptyset$.

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They also have various applications:

In practise: image compression (Barnsley *et al.*).

In theory: dimensions of self-similar sets and measures with complicated overlaps (Testud, Olsen, Snigireva).

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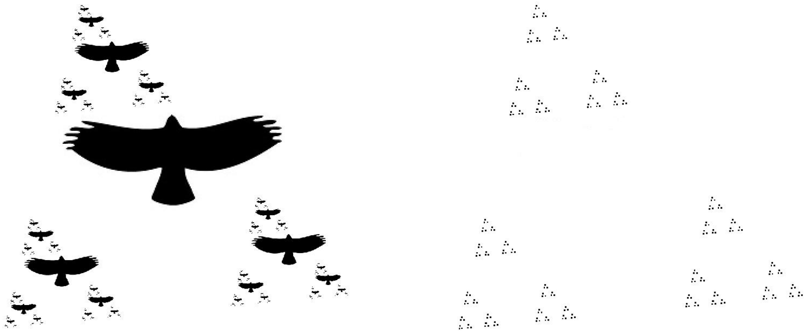

Figure : *A flock of birds from above*

Inhomogeneous iterated function systems - examples

Figure : *A fractal forest*

Dimension theory of homogeneous attractors

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When considering the dimension 'dim' of F_C, one expects the relationship

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\end{aligned}$$

and so the expected formula holds trivially.

Box dimensions of inhomogeneous self-similar sets

The upper and lower box dimensions are **not countably stable** and so establishing the relationships

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can be awkward.

Although the initial philosophy was that we should still expect them to hold.

Box dimensions of inhomogeneous self-similar sets

Theorem (Olsen-Snigireva 2007)

If the ambient metric space is a subset of $\mathbb{R}^d$, each of the S_i are similarities, and the sets $S_1(F_C), \dots, S_N(F_C)$ and C are pairwise disjoint, then

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Remark

The above result was obtained as a corollary to deeper result concerning the L^q-dimensions of inhomogeneous self-similar measures.

Box dimensions of inhomogeneous self-similar sets

Theorem (F. 2012)

Working in an arbitrary compact metric space, still assuming each of the S_i are similarities, but with no assumptions on separation conditions, we have

$$\max\{\overline{\dim}_B F_\emptyset,\ \overline{\dim}_B C\} \leqslant \overline{\dim}_B F_C \leqslant \max\{s,\ \overline{\dim}_B C\}$$

where s is the similarity dimension.

Corollary (F. 2012)

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(1) *The SOSC is satisfied - this still allows for arbitrary overlaps concerning C.*

(2) *The ambient metric space is a subset of $\mathbb{R}^d$ and the OSC is satisfied.*

(3) *The ambient metric space is a subset of $\mathbb{R}$, the defining parameters for the IFS are algebraic and the semigroup generated by the maps is free.*

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We provide simple examples of this failure where the ambient space is $[0,1]^d$ and one can assume as strong separation conditions as one wishes.

We also provide (slightly unsightly) upper and lower bounds on $\underline{\dim}_B F_C$ which hold generally when the ambient metric space is Ahlfors regular and some separation properties are assumed for the underlying IFS.

Corollary (F. 2012)

Even in the simplest setting, $\underline{\dim}_B F_C$ cannot be given as a function of the upper and lower box dimensions of $F_\emptyset$ and C.

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The lower box dimension of inhomogeneous attractors is difficult to study!

Self-affine carpets

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Perhaps inhomogeneous versions of the Bedford-McMullen carpets will provide interesting examples and different phenomena?

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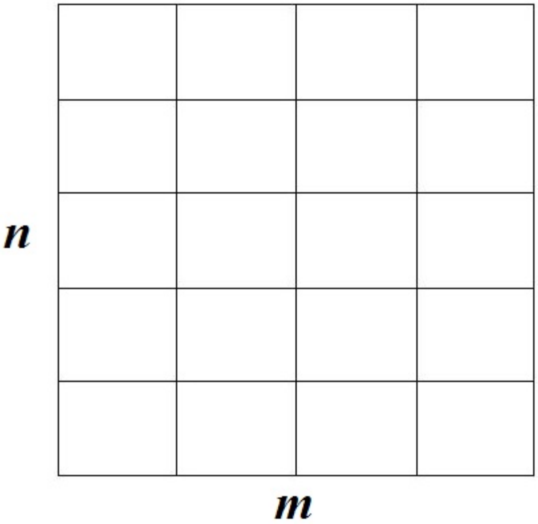

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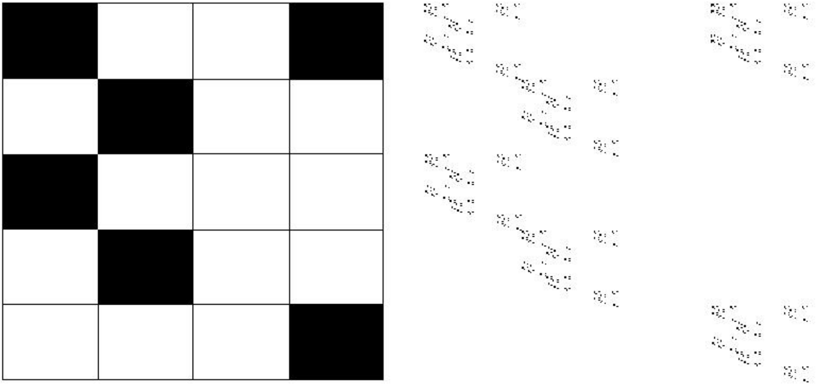

Figure : A self-affine Bedford-McMullen carpet with $m = 4$, $n = 5$. The shaded rectangles on the left indicate the 6 maps in the IFS.

Box dimensions of inhomogeneous self-affine carpets

Let π_1 denote the orthogonal projections from the plane onto the first coordinates and write

$$s_1(F_\emptyset) = \dim_{\mathrm{B}} \pi_1(F_\emptyset)$$

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Let N be the number of mappings in the IFS.

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Theorem (Bedford-McMullen 1985)

For a homogeneous Bedford-McMullen carpet $F_\emptyset$, we have

$$\overline{\dim}_B F_\emptyset = \underline{\dim}_B F_\emptyset = \frac{\log N}{\log n} + s_1(F_\emptyset)\left(1 - \frac{\log m}{\log n}\right)$$

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Theorem (F 2013)

For an inhomogeneous Bedford-McMullen carpet F_C, we have

$$\overline{\dim}_B F_C = \frac{\log N}{\log n} + \max\{s_1(F_\emptyset), \overline{s}_1(C)\}\left(1 - \frac{\log m}{\log n}\right)$$

assuming a 'regularity condition' on C.

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Box dimensions of inhomogeneous self-affine carpets

- we also have non-trivial estimates on the lower box dimension of F_C.
- our results actually apply to much more general families of carpet than Bedford-McMullen, for example Lalley-Gatzouras and Barański.

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The IFS is then made up of all the maps which correspond to the left hand column.

The condensation set for this construction is the base of the unit square.

We call the inhomogeneous attractor a *fractal comb* and denote it by F_C^n.

Fractal combs

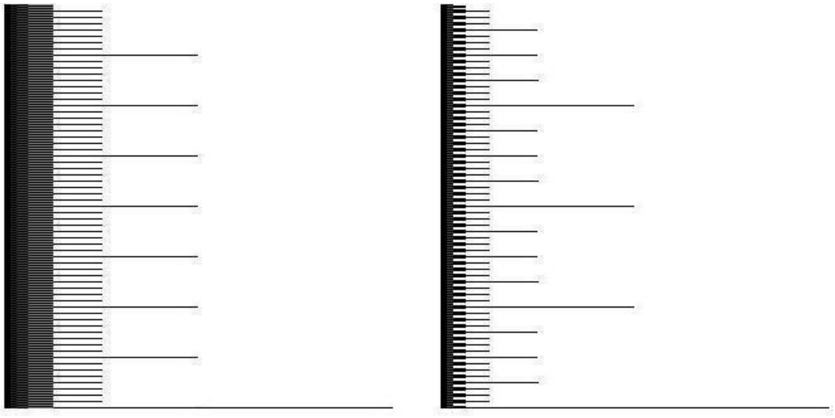

Figure : The inhomogeneous fractal combs F_C^8 (left) and F_C^4 (right).

Fractal combs

Our results imply that

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$$= 2 - \log 2 / \log n > 1.$$

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However,

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However,

$$\max\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\; \overline{\dim}_{\mathrm{B}} C\} \;=\; 1$$

Corollary (F. 2013)

In the case of inhomogeneous Bedford-McMullen carpets, $\overline{\dim}_B F_C$ cannot be given as a function of the upper and lower box dimensions of $F_\emptyset$ and C. In particular, it depends on the IFS.

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Theorem (F 2013)

For any inhomogeneous Bedford-McMullen carpet F_C

$$\overline{\dim}_B F_C \geqslant \frac{\log N}{\log n} + \max\{s_1(F_\emptyset), \overline{s}_1(C)\} \left(1 - \frac{\log m}{\log n}\right)$$

but the inequality can be strict.

Future work

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Does the 'expected result' hold for upper box dimension for every inhomogeneous self-similar set, even if there is a dimension drop in the homogeneous analogue?

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Question

What about more general self-affine constructions? Is there an inhomogeneous version of Falconer's Theorem?

Thank you!

Main references

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