# Generic dimensions of graphs and images

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*joint work with R. Balka, K. Falconer, Á. Farkas and J. Hyde*

# Graphs and images

Let $X$ be a compact metric space, $n \in \mathbb{N}$ and write

$$C_n(X) = \Big\{ f : X \to \mathbb{R}^n \mid f \text{ is continuous} \Big\}.$$

This is a Banach space when equipped with the infinity norm, $\| \cdot \|_\infty$.

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We interested in studying two objects related to a given $f \in C_n(X)$. The image:

$$f(X) \subset \mathbb{R}^n$$

and the graph:

$$G_f = \Big\{ (x, f(x)) \mid x \in X \Big\} \subset X \times \mathbb{R}^n.$$

# Generic dimension of graphs of continuous functions

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Clearly, this question can mean different things depending on the definition of the words 'dimension' and 'generic'!

# Dimension

There are, of course, several different notions of 'dimension' used to study fractal sets. Some of the most widely used include Hausdorff dimension, packing dimension and box-counting dimension. These are related in the following way for an arbitrary totally bounded set $K$.

$$\begin{tikzcd}
 & \dim_{\mathrm{P}} K \arrow[dr, phantom, sloped, "\leqslant"] & \\
\dim_{\mathrm{H}} K \arrow[ur, phantom, sloped, "\leqslant"] \arrow[dr, phantom, sloped, "\leqslant"] & & \overline{\dim}_{\mathrm{B}} K \\
 & \underline{\dim}_{\mathrm{B}} K \arrow[ur, phantom, sloped, "\leqslant"] &
\end{tikzcd}$$

# How should we define ‘generic’?

In mathematics one is often interested in making statements about a ‘generic’ member of some family. (*Almost all* real numbers are *normal*, for example.) It is therefore important to develop a rigorous framework in which a sensible definition of ‘generic’ can be given. We will focus on two major approaches to this problem:

(1) Prevalence;

(2) Typicality.

# Prevalence: a measure theoretic approach

### Definition

*Let $X$ be a completely metrizable topological vector space. A Borel set $F \subseteq X$ is prevalent if there exists a Borel measure $\mu$ on $X$ and a compact set $K \subseteq X$ such that $0 < \mu(K) < \infty$ and*

$$\mu\Big(X \setminus F + x\Big) = 0$$

*for all $x \in X$.*

*A non-Borel set $F \subseteq X$ is prevalent if it contains a prevalent Borel set and the complement of a prevalent set is called a shy set.*

# Prevalence: an extension of ‘Lebesgue almost all’ to infinite dimensional spaces

Prevalence was introduced by Hunt, Sauer and Yorke in 1992. The importance of prevalence is that it extends the notion of ‘Lebesgue almost all’ to infinite dimensional spaces where there is no Lebesgue measure. It satisfies many of the natural properties one would want from a definition of ‘generic’. For example:

(1) A superset of a prevalent set is prevalent;

(2) Prevalence is translation invariant;

(3) A countable intersection of prevalent sets is prevalent;

(4) In *finite* dimensional vector spaces prevalent sets are precisely the sets with full Lebesgue measure.

# Typicality: a topological approach

### Definition

*Let $X$ be a complete metric space. A set $M$ is called meagre if it can be written as a countable union of nowhere dense sets. A property is called typical if the set of points which do not have the property is meagre.*

Perhaps surprisingly, typicality often completely disagrees with the measure theoretic approach to describing generic behaviour. For example, a typical real number is not normal.

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# Extensions to $C_1(X)$ in the typicality case

In fact, Hyde, Laschos, Olsen, Petrykiewicz and Shaw proved a much more general result.

**Theorem (Hyde, Laschos, Olsen, Petrykiewicz and Shaw '10)**

*Let $X$ be compact. Then for the typical $f \in C_1(X)$ we have*

$$\underline{\dim}_{\mathrm{B}} G_f = \underline{\dim}_{\mathrm{B}} X$$

*and*

$$\overline{\dim}_{\mathrm{B}} G_f = \sup_{g \in C_1(X)} \overline{\dim}_{\mathrm{B}} G_g.$$

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$$\overline{\dim}_{\mathrm{B}} G_f = \sup_{g \in C_1(X)} \overline{\dim}_{\mathrm{B}} G_g .$$

The number $\sup_{g \in C_1(X)} \overline{\dim}_{\mathrm{B}} G_g$ is called the *graph upper box dimension* of $X$ and can be difficult to compute explicitly.

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Finally, in 2011 it was shown by F and Hyde that the prevalent Hausdorff dimension is also 2.

$$\dim_{\mathrm{H}} G_f = \underline{\dim}_{\mathrm{B}} G_f = \dim_{\mathrm{P}} G_f = \overline{\dim}_{\mathrm{B}} G_f = 2$$

# Extensions to $C_1(X)$ in the prevalence case

In 2011, Bayart and Heurteaux generalised the result of F and Hyde to the following theorem.

**Theorem (Bayart and Heurteaux '11)**

*Suppose that* $\dim_{\mathrm{H}} X > 0$*. Then the set*

$$\{f \in C_1(X) \mid \dim_{\mathrm{H}} G_f = \dim_{\mathrm{H}} X + 1\}$$

*is a prevalent subset of* $C_1(X)$*.*

The key technique in the proof was to use fractional Brownian motion on $X$ and the assumption $\dim_{\mathrm{H}} X > 0$ was required to guarantee the existence of certain measures on $X$ which could be lifted to the graphs and used in the energy estimates. Interestingly, this left open the case when $\dim_{\mathrm{H}} X = 0$.

# Dimension of prevalent images in $C_n(X)$

### Theorem (Dougherty '94)

*Let $X$ be homeomorphic to the Cantor space. Then for the prevalent $f \in C_n(X)$, the image $f(X)$ has non-empty interior.*

In 2012, this result was applied by Balka, Farkas, F and Hyde to obtain the following Corollary.

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In 2012, this result was applied by Balka, Farkas, F and Hyde to obtain the following Corollary.

### Corollary (Balka, Farkas, F and Hyde '12)

*Let $X$ be uncountable. Then the set*

$$\{f \in C_n(X) \mid \dim_{\mathrm{H}} f(X) = n\}$$

*is a prevalent subset of $C_n(X)$.*

# Dimension of prevalent images in $C_n(X)$

This result has a surprising application. The image $f(X)$ is the projection of the graph $G_f$ onto $\mathbb{R}^n$ and hence, for all $f \in C_n(X)$,

$$\dim_{\mathrm{H}} f(X) \leqslant \dim_{\mathrm{H}} G_f \leqslant \dim_{\mathrm{H}}(X \times \mathbb{R}^n) \leqslant \dim_{\mathrm{H}} X + n.$$

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Now, if we suppose $\dim_{\mathrm{H}} X = 0$, then we obtain

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and so, for the prevalent $f \in C_n(X)$,

$$\dim_{\mathrm{H}} G_f = n$$

which solves the open question left by Bayart and Heurteaux when we set $n = 1$. Note that if $X$ is countable, then the problem is trivial.

# Dimension of typical images in $C_n(X)$

There is a substantial amount of literature on the problem of computing typical dimensions in a wide variety of contexts. A 'rule of thumb' is that the typical Hausdorff dimension tends to be the smallest value it can be and the typical packing dimension tends to be the largest value it can be.

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and the largest the dimension of $f(X)$ can be for $f \in C_n(X)$ is $n$ - this is slightly less easy to see. In fact, every uncountable compact metric space contains a Cantor set and every compact set is a continuous image of a Cantor set. These two facts, combined with Tietze's extension Theorem do the job.

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NB: the constant maps cannot form a residual set, so perhaps something more exciting is happening for Hausdorff dimension?!

# Some simple cases

Clearly, working in the space $C_1([0,1])$, we have for the typical $f$

$$\dim_{\mathrm{H}} f([0,1]) \;=\; \dim_{\mathrm{P}} f([0,1]) \;=\; 1.$$

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So Hausdorff dimension is not typically minimized! We also have that for a typical $f \in C_n([0,1])$

$$\dim_{\mathrm{H}} f([0,1]) \; = \; 1 \qquad \text{and} \qquad \dim_{\mathrm{P}} f([0,1]) \; = \; n.$$

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The typical packing dimension is maximal as expected, but what is the typical Hausdorff dimension of $f(X)$ for general $X$?

# Topological dimension and an embedding theorem

### Definition

*The topological dimension of a topological space $X$ is defined to be the maximum value of $n$, such that there exists $\varepsilon > 0$ such that for all $\delta < \varepsilon$ and all $\delta$-covers of $X$, there exists a point $x \in X$ which lies in at least $n + 1$ of the covering sets.*

*We denote this number by* $\dim_{\mathrm{T}} X$ *and note that it is always an integer.*

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*We denote this number by* $\dim_{\mathrm{T}} X$ *and note that it is always an integer.*

For any metric space, $X$, we have

$$\dim_{\mathrm{T}} X \; \leqslant \; \dim_{\mathrm{H}} X \; \leqslant \; \dim_{\mathrm{P}} X$$

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**Theorem (Balka, Farkas, F, Hyde '12)**

*For a typical $f \in C_n(X)$ we have*

$$\dim_{\mathrm{H}} f(X) \;=\; \underline{\dim}_{\mathrm{B}} f(X) \;=\; \min\left\{ \dim_{\mathrm{T}} X,\; n\right\}$$

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*and*

$$\dim_{\mathrm{P}} f(X) \;=\; \overline{\dim}_{\mathrm{B}} f(X) \;=\; n.$$

The typical Hausdorff dimension is not minimal (or maximal) but yet it is still an integer, even if the Hausdorff dimension of $X$ is fractional!

# Further work

One might hope to generalise many of the results presented here. In particular, in the graph setting, one could consider the space $C_n(X)$.

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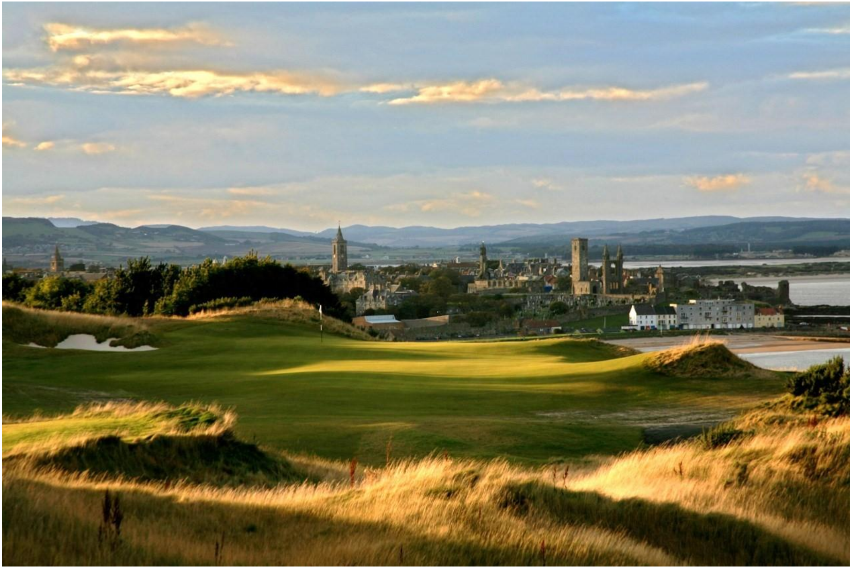
Also, one might try to map into a more general space that $\mathbb{R}^n$, for example, an arbitrary Banach space. Clearly there will be some difficulties, but perhaps one can say something in some specific cases?

Thank you!



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