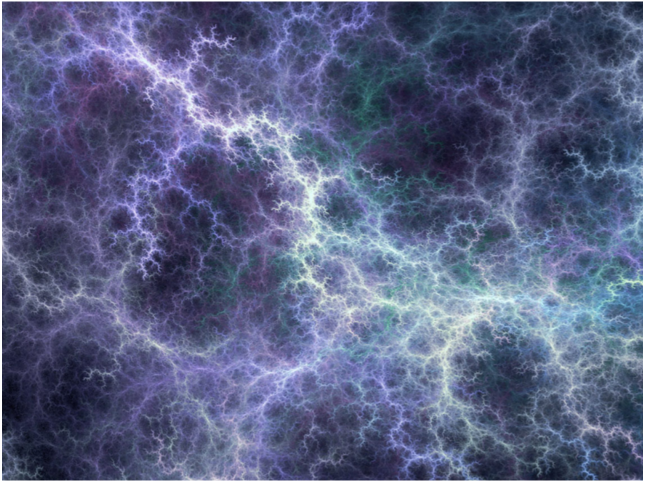
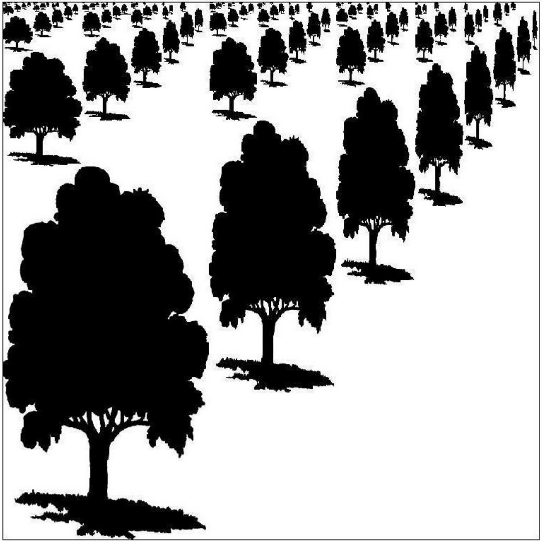
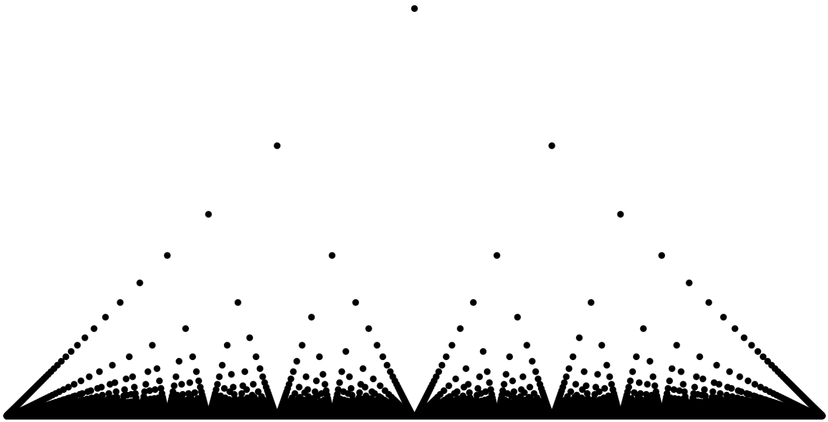
Fractals in science and nature

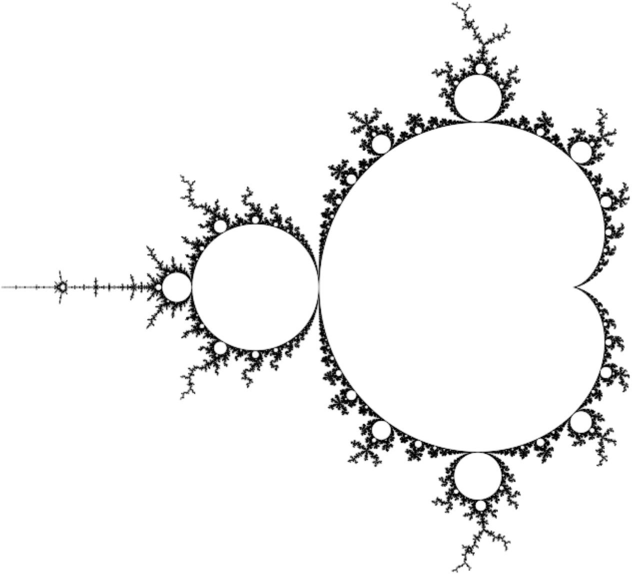
Jonathan Fraser

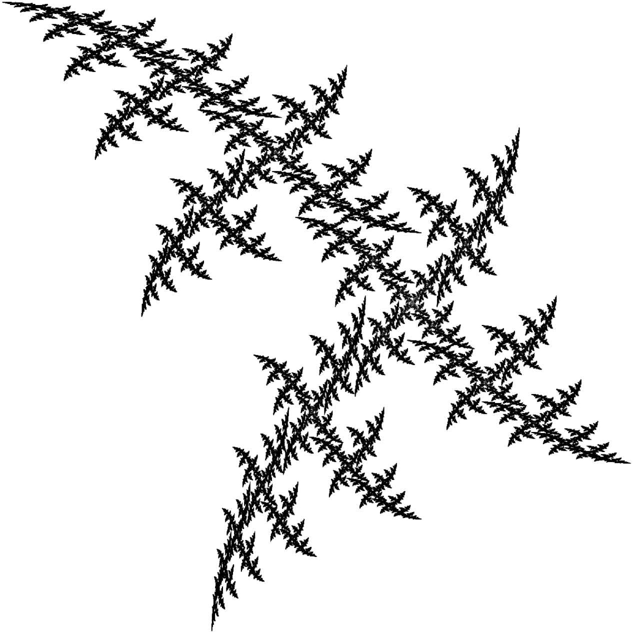
University of St Andrews











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- complexity?
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- a “natural” look?
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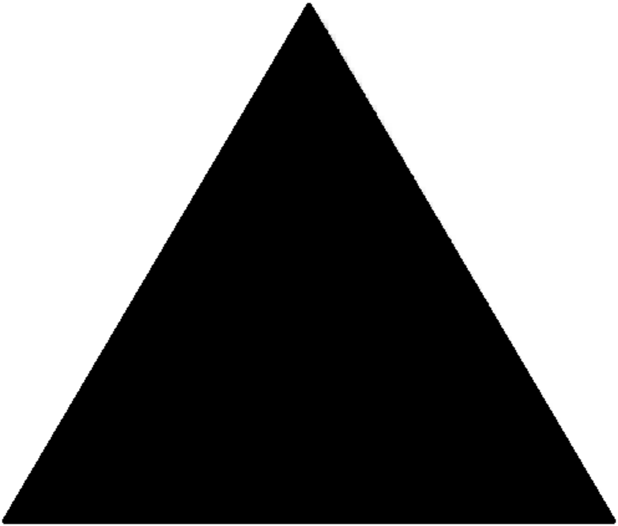
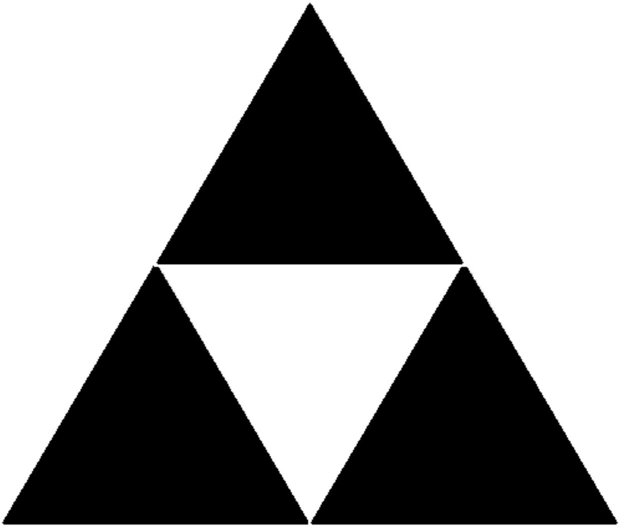
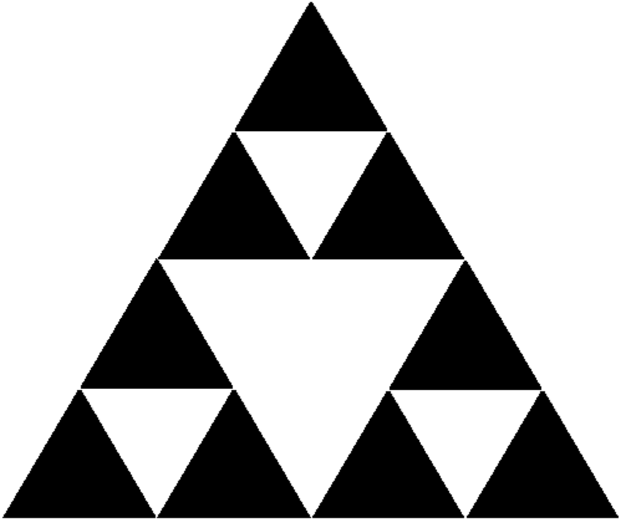
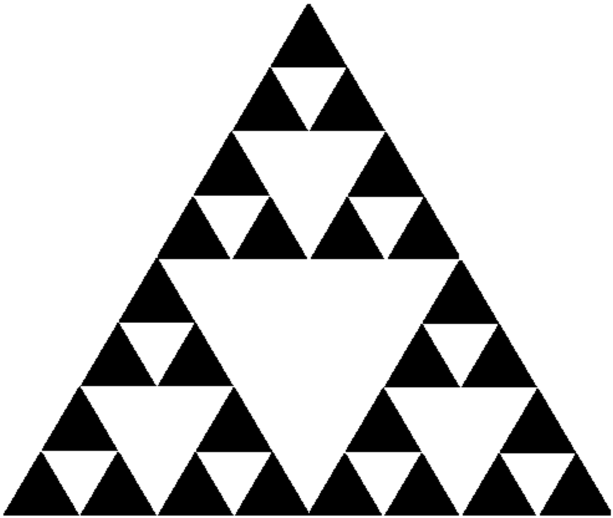
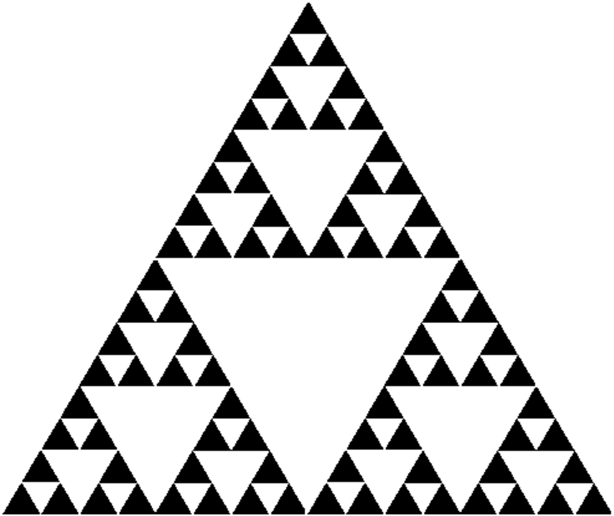
- detail at a fine scale?

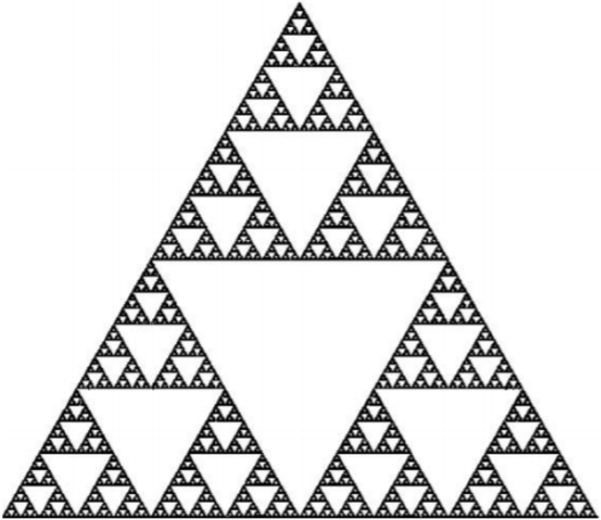












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- stock market fluctuations
- the surface of a lung
- horizons of mountain landscapes
- distribution of stars in the galaxy

Where do circles, lines and triangles appear in the real world?

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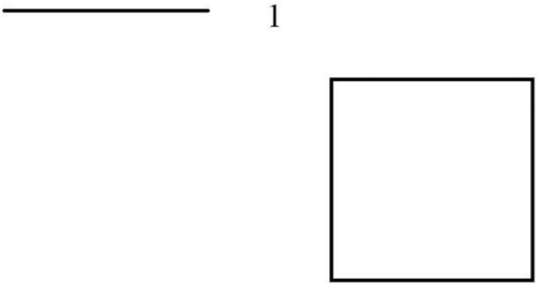
actually, they don't.


1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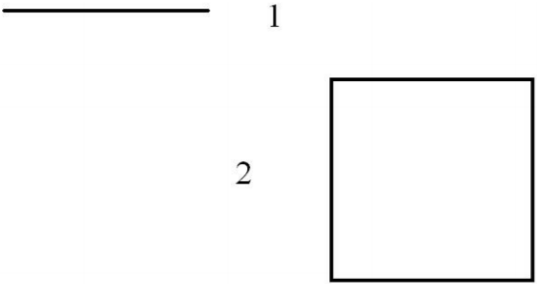
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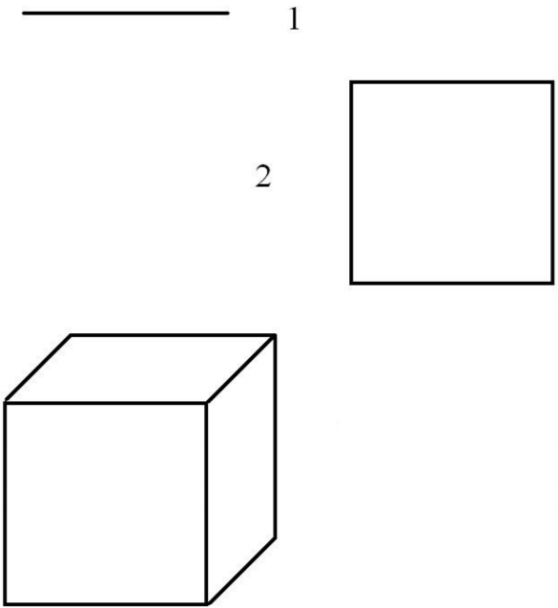
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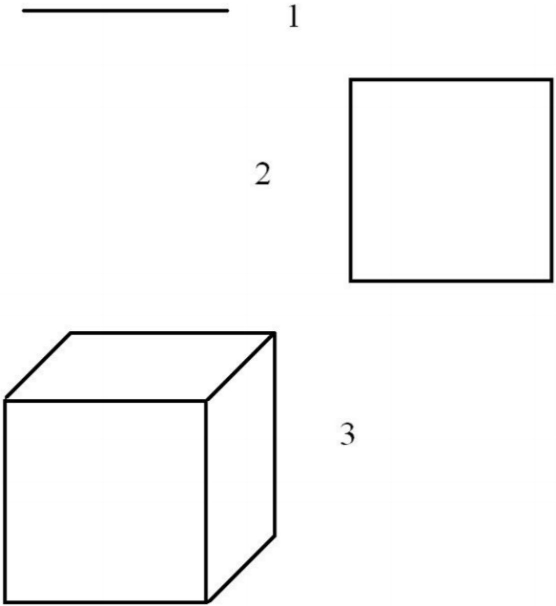
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Can we define the dimension of a fractal?

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What does “dimension” mean?

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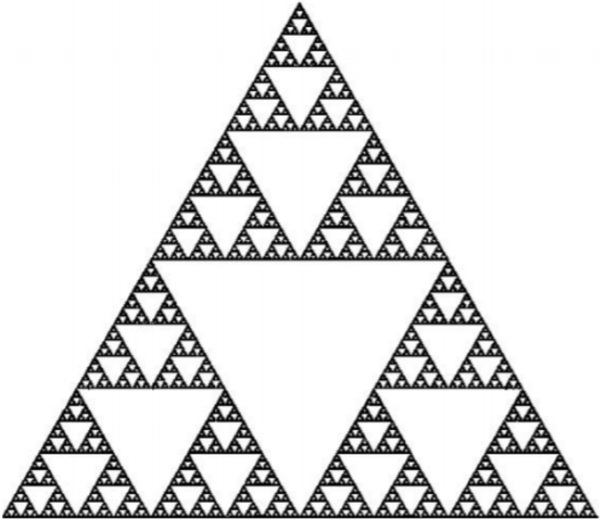
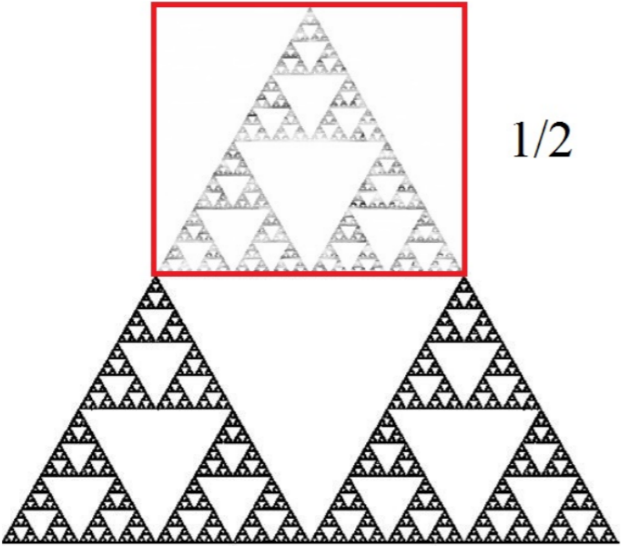
So, perhaps $N(r) \approx r^{-\text{dimension}}$ in general?



Roughly how many small squares of sidelength $r > 0$ to we need to cover the fractal Sierpiński triangle?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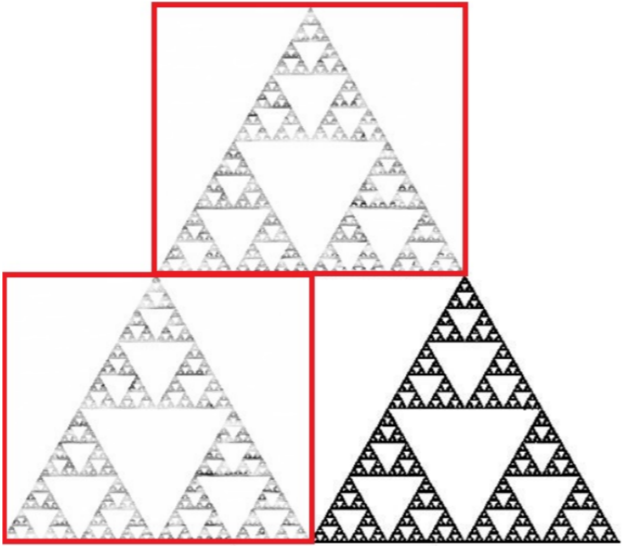

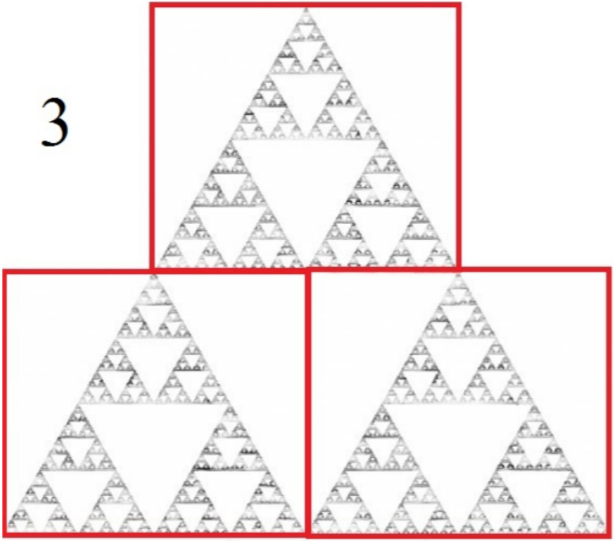
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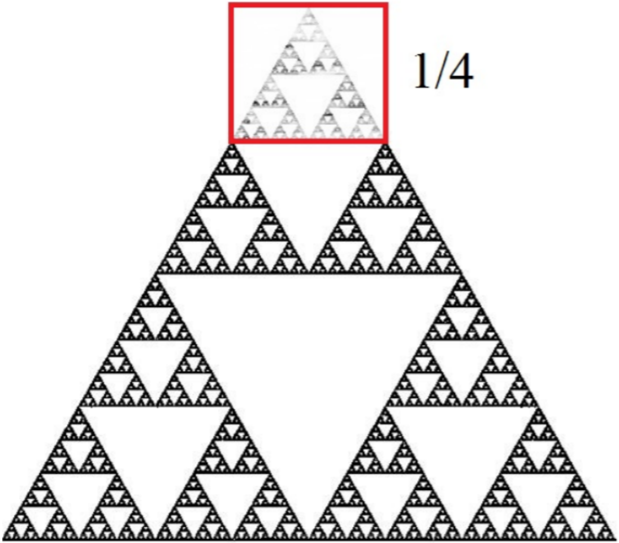
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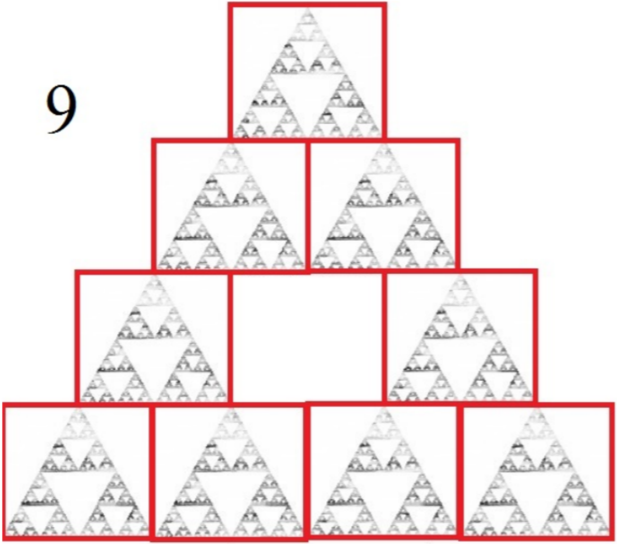
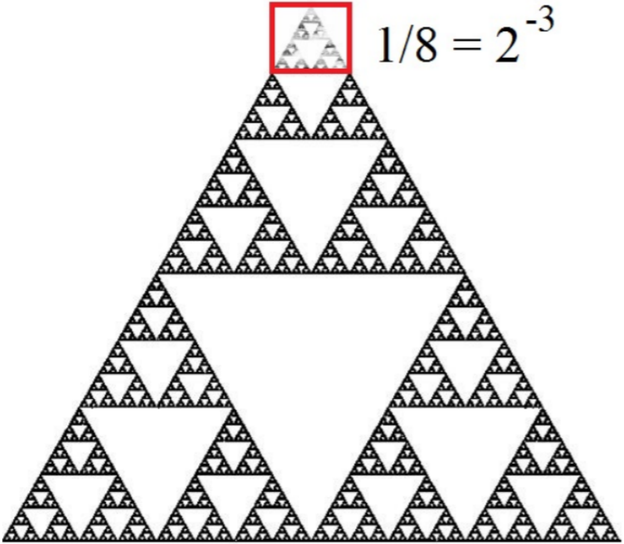
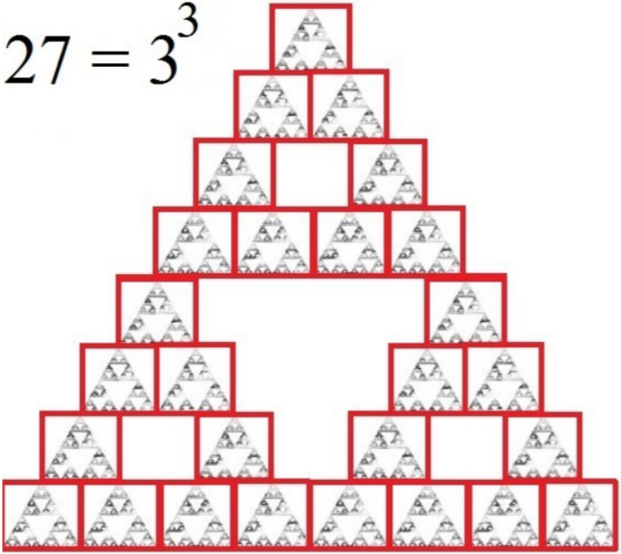




$1/8 = 2^{-3}$

$$27 = 3^3$$



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The dimension of the Sierpiński triangle is $\log_2(3) \approx 1.5849625\ldots$

Thank you for listening!

Figure: 'Circle Limit III' by M.C. Escher