Dynamically defined fractals

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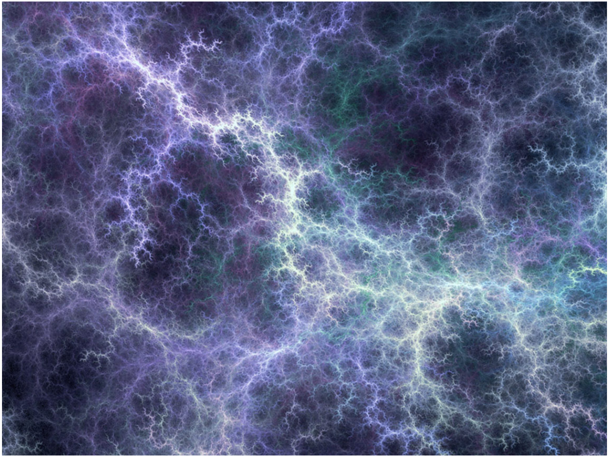
What is a fractal?

Some examples

Fractals

Fractals



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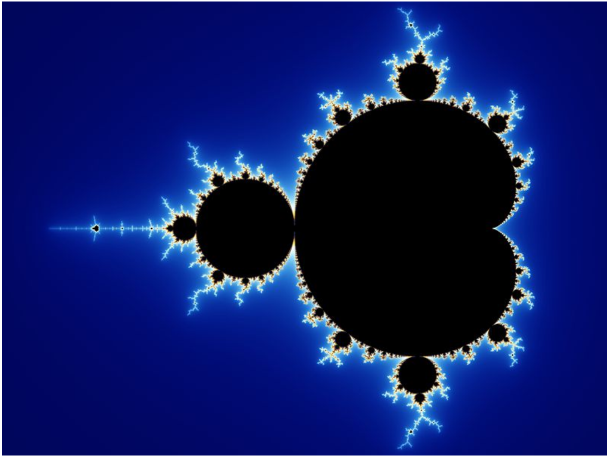

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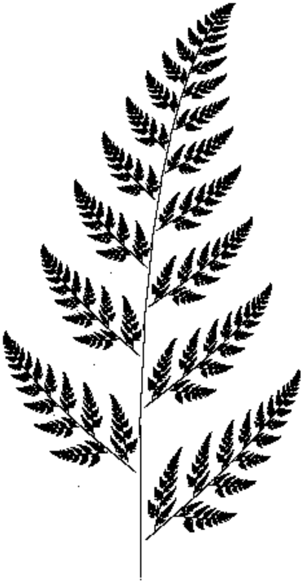

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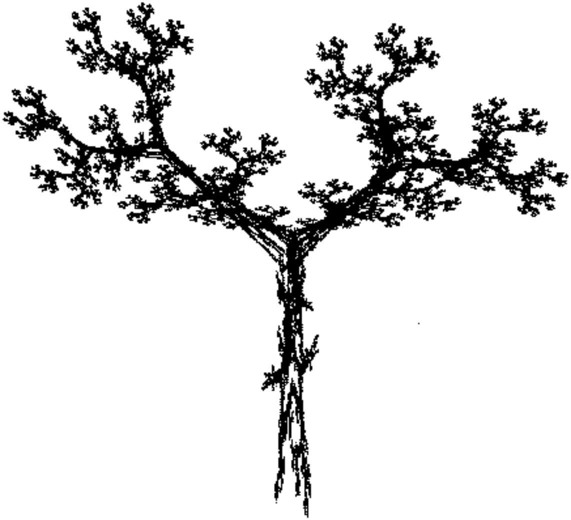

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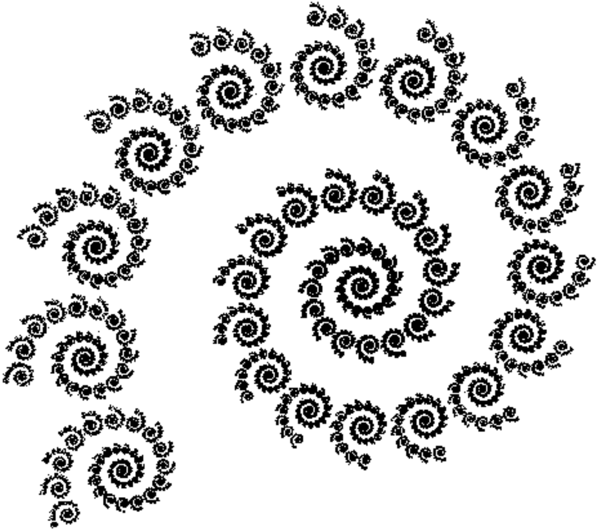

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Fractal geometry is the study of fractals and is mainly concerned with examining their geometrical properties in a rigorous framework.

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(I guess you know)

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(4) Is the system 'mixing'? How fast does it 'mix'?

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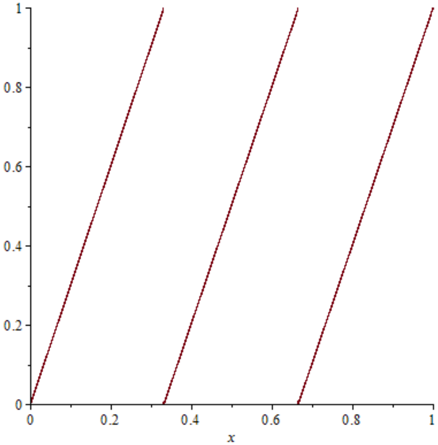

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i.e. T acts like the left shift on the trinary expansion of x.

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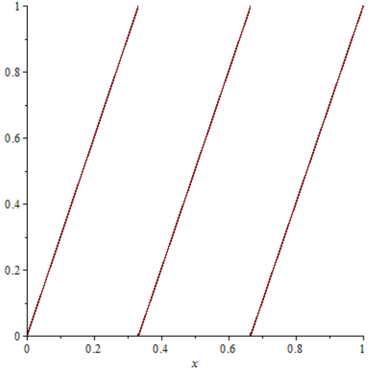

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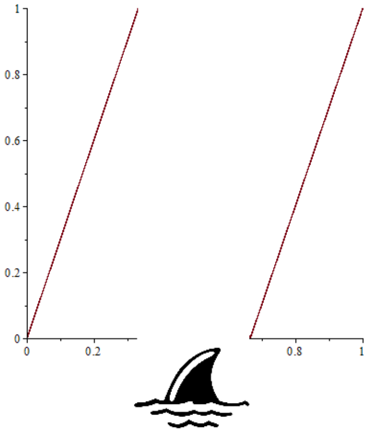

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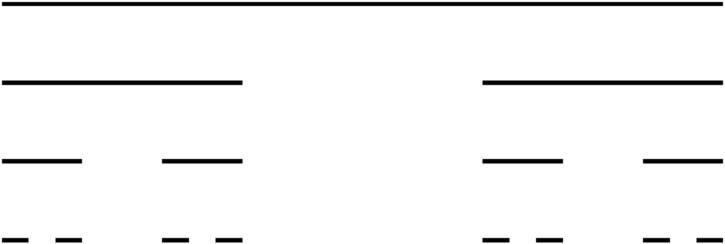

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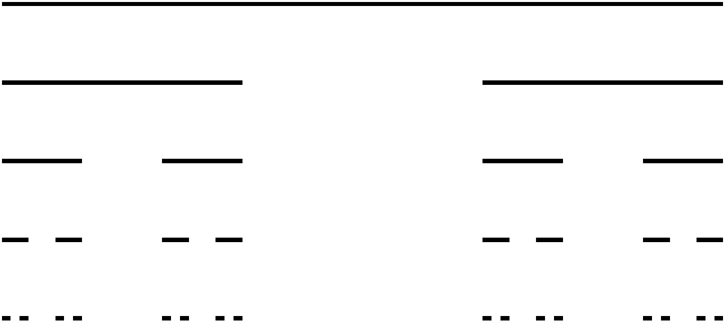

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Do we expect the 'density'

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This time the answer is ‘almost surely yes’, and this is an easy consequence of the underlying dynamics.

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and so either $D(x) < \lambda$ almost surely or $D(x) \geqslant \lambda$ almost surely.

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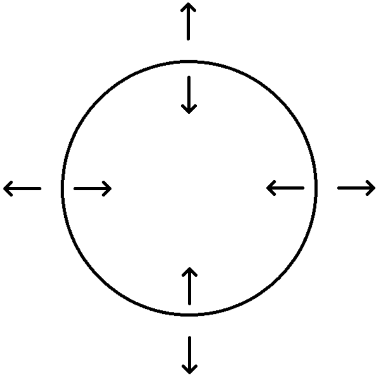
For $c = 0$



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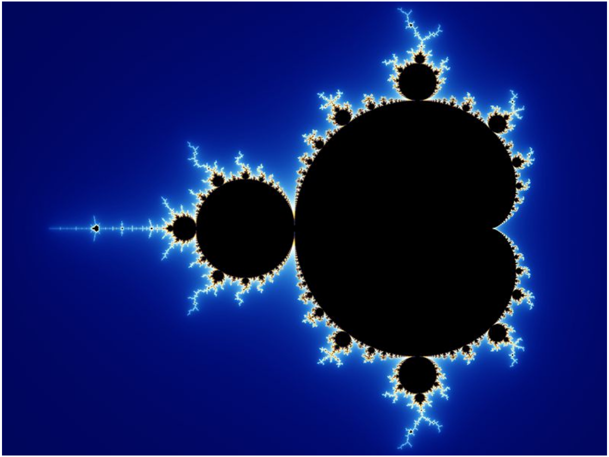
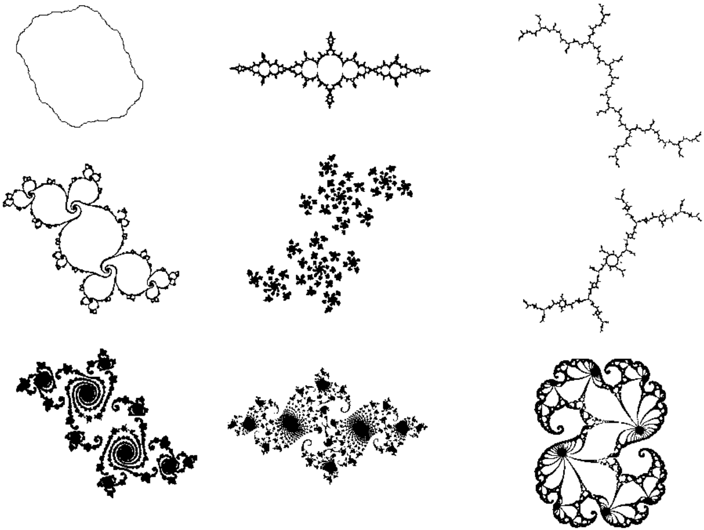
The Mandelbrot set (which we've seen already) is

$$M \ = \ \{c \in \mathbb{C} \ : \ J(T) \text{ is connected}\}.$$

The Mandelbrot set



Some Julia sets



Basic properties of the Julia set J

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(5) T acts 'chaotically' on J

Other examples

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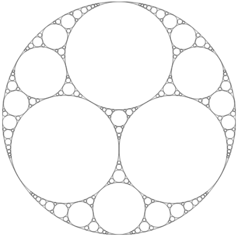
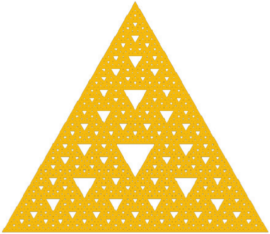
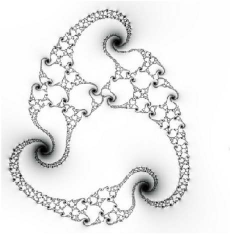
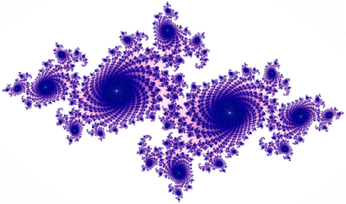
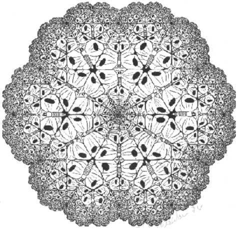
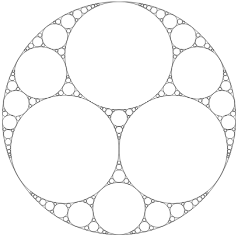
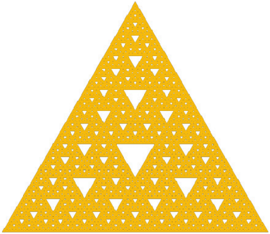
Iterated function systems are collections of contracting maps on a metric space. They uniquely define non-empty compact fractal attractors which can be modelled by shift spaces similar to the Cantor set.







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