Fourier transforms of measures on the Brownian graph

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ICERM
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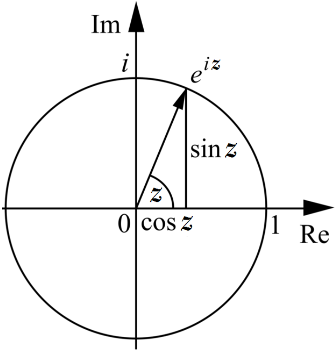
My co-authors

Fourier transforms and dimension

The **Fourier transform** of a measure μ on $\mathbb{R}^d$ is a function $\hat{\mu} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\hat{\mu}(x) = \int \exp(-2\pi i\, x \cdot y)\, d\mu(y).$$



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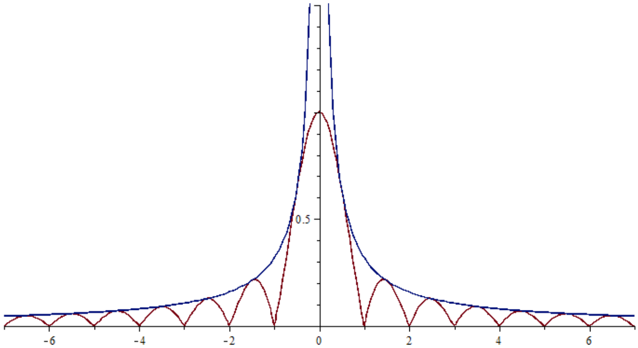
$$|\hat{\mu}(x)| \lesssim |x|^{-s/2} \;\Rightarrow\; I_{s-}(\mu) < \infty \;\Rightarrow\; \dim_{\mathrm{H}} K \geqslant s$$

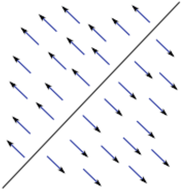
Simple example

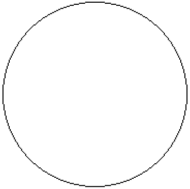
For example, if μ is Lebesgue measure on the unit interval, then a quick calculation reveals that for $x \in \mathbb{R}$

$$|\hat{\mu}(x)| \;=\; \left| \int_0^1 \exp\left(-2\pi ixy\right) dy \right| \;\leqslant\; \frac{1}{\pi}\;|x|^{-1}.$$



Examples with no decay





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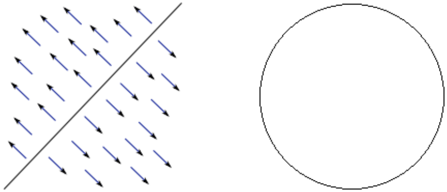

The middle 3rd Cantor set also supports no measures with Fourier decay!

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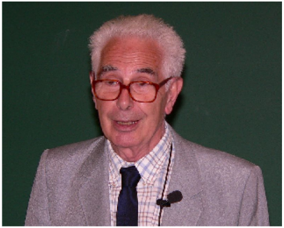
$$\dim_{\mathrm{F}} K \leqslant \dim_{\mathrm{H}} K$$

Sets with equality are called **Salem sets**.

Salem 1898-1963



Kahane 1926-

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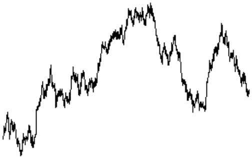

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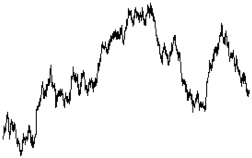

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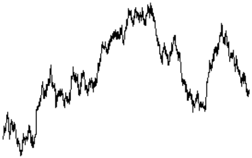

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- It remained open for a long time whether or not graphs are almost surely Salem.

- Kahane explicitly asked the question in 1993 (also asked by Shieh-Xiao in 2006).

Our work on Brownian motion

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The graph of Brownian motion is almost surely **not** *a Salem set. In fact, there does not exist a measure* μ *supported on a graph which satsfies:*

$$|\hat{\mu}(x)| \lesssim |x|^{-s/2}$$

for any $s > 1$. *Therefore* $\dim_{\mathrm{F}} G(B) \leqslant 1 < 3/2 \overset{a.s.}{=} \dim_{\mathrm{H}} G(B)$.

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- **Key idea**: we proved a **new slicing theorem** for planar sets supporting measures with fast Fourier decay.

- the answer to Kahane's problem is geometric (not stochastic).

Slicing theorems

Theorem (Marstrand)

Suppose $K \subset \mathbb{R}^2$ *has* $\dim_{\mathrm{H}} K > 1$*, then for almost all directions* $\theta \in S^1$*, Lebesgue positively many* $y \in \mathbb{R}$ *satisfy:*

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Theorem (F-Sahlsten, 2015)

Let μ be the push forward of Lebesgue measure on the graph of Brownian motion. Then almost surely

$$|\hat{\mu}(x)| \lesssim |x|^{-1/2}\sqrt{\log|x|} \qquad (x \in \mathbb{R}^2)$$

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This time our proof was stochastic (not geometric) and relied on techniques from **Itô calculus**, as well as adapting some of Kahane's ideas.

Sketch proof

Write $x \in \mathbb{R}^2$ in polar coordinates as

$$x = (u\cos\theta,\ u\sin\theta)$$

and observe that

$$\hat{\mu}(x) \ = \ \int_0^1 \overbrace{\exp\left(-2\pi i u(t\cos\theta + B(t)\sin\theta)\right)}^{Z_t} dt.$$

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We must prove

$$|\hat{\mu}(x)| \ \overset{a.s.}{\leqslant} \ Cu^{-1/2}\sqrt{\log u}$$

with C independent of θ.

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where T is chosen such that $Z_T = Z_0 = 1$ and the interval $(0, T)$ contains all 'full rotations'. Similar to Lebesgue case:

$$\left| \int_T^1 Z_t dt \right| \lesssim u^{-1/2}$$

so it remains to consider the 'full rotations'.

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• use Kahane's techniques to transform moment estimates back to an almost sure estimate for the Fourier transform

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Thanks!!

Brownie in motion