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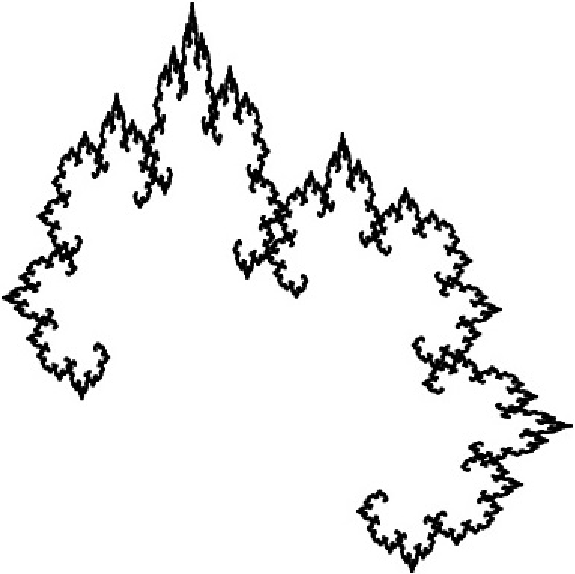
Here $N_r(E)$ is the minimum number of balls of radius r required to cover a set E.

Dimension theory



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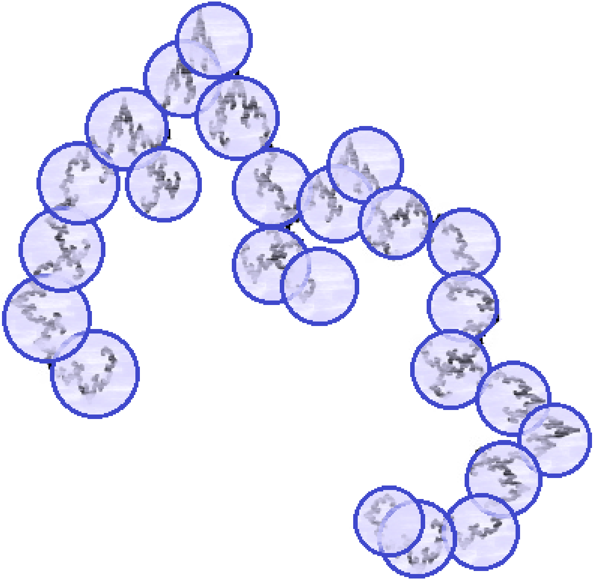

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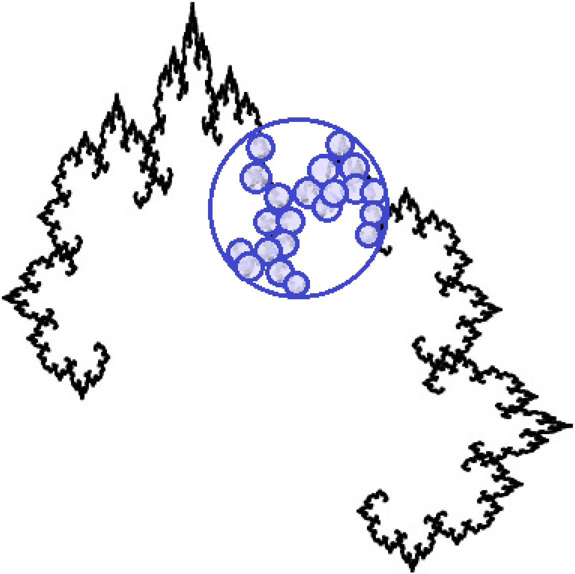

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$$\dim_{\mathrm{H}} F \leq \dim_{\mathrm{P}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}} F.$$

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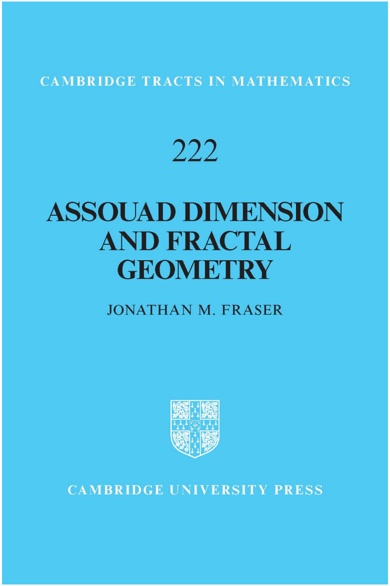

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$$\dim_{\mathrm{H}} F > 5/4 \Rightarrow \dim_{\mathrm{H}} D(F) = 1.$$

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Theorem (F 2020)

For an arbitrary set $F \subseteq \mathbb{R}^2$

$$\dim_A D(F) \geq \min\{\dim_A F, 1\}$$

and this is sharp.

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The case $\dim_{\mathrm{A}} F \leq 1$ is not (so far) susceptible to such reductions.

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Sketch proof: For $V \in G(2,1)$ write Π_V for orthogonal projection onto V. For $z \in \mathbb{R}^2$, write π_z for the associated radial projection and D_z for the pinned distance map.

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Suppose F is closed and let $E \in \mathrm{Micro}(F)$ with $\dim_{\mathrm{H}} E = \dim_{\mathrm{A}} F$. Let $E' \in \mathrm{Micro}(E)$ with $\dim_{\mathrm{H}} E' = \dim_{\mathrm{A}} E = \dim_{\mathrm{A}} F$ with 'focal point' $z \in E$.

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Let $\mathcal{E} \subseteq G(2,1)$ be the set of exceptions to Orponen's projection theorem for Assouad dimension applied to the set E'. That is $\mathcal{E} \subseteq G(2,1)$ are those V for which $\dim_{\mathrm{A}} \pi_V E' < \min\{\dim_{\mathrm{A}} E', 1\}$. Orponen's theorem (2021) states that $\dim_{\mathrm{H}} \mathcal{E} = 0$.

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and therefore

$$\dim_{\mathrm{A}} D(F) \geq \dim_{\mathrm{A}} D(E) \geq \dim_{\mathrm{A}} D_z(E) \geq \dim_{\mathrm{H}} E = \dim_{\mathrm{A}} F.$$


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Thanks!!