Scaling scenery and the distance set problem

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joint with T. Sahlsten and A. Ferguson and with M. Pollicott

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Question: Suppose E has cardinality n, what is the minimum possible cardinality of $D(E)$?

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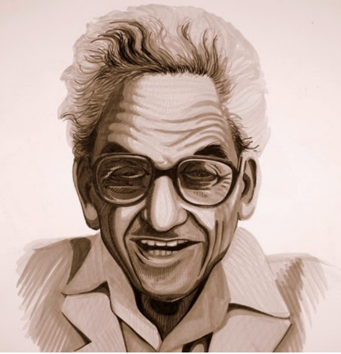
Theorem (Erdös 1946)

For sets E in the plane

$$\sqrt{n - 3/4} - 1/2 \; \leqslant \; g(n) \; \leqslant \; cn/\sqrt{\log n}$$



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• But the Landau-Ramanujan Theorem says that the number of positive integers less than x that are the sum of two squares is bounded by a constant times $x/\sqrt{\log x}$.

• So $|D(E)|$ is bounded above by a constant times $n/\sqrt{\log n}$

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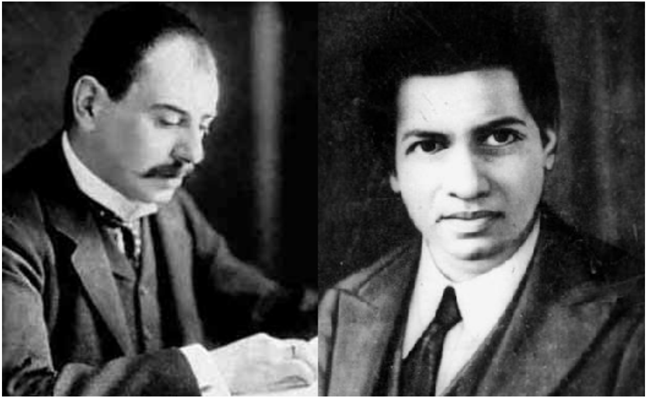

Landau and Ramanujan

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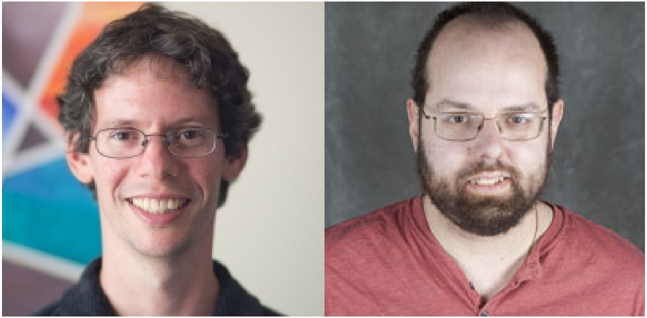

Larry Guth and Netz Hawk Katz

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- 2003 Bourgain: there exists a constant $c > 1/2$, such that for planar E

$$\dim_{\mathrm{H}} E > 1 \Rightarrow \dim_{\mathrm{H}} D(E) > c$$

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• Orponen's proof was split into two cases:

1) 'Dense rotations': recent pioneering work of Hochman and Shmerkin on fractal projection theorems gives the desired result.

2) 'Discrete rotations': This case can be reduced to the 'no rotations case', and from there a delicate geometric argument yields the result.

• Mike Hochman and Pablo Shmerkin published an important paper in 2012 which studied the ergodic theory of the process of 'blowing up' a measure.

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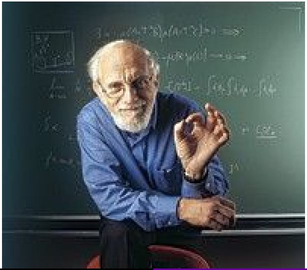
• Ideas date back to Hillel Furstenberg in the 60s-70s, but rediscovered recently by Furstenberg (2008), Gavish (2011), Hochman-Shmerkin (2012) and Hochman (2010/2013).



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- If $k \in \mathbb{N}$, let $D_k(x) \in \mathcal{D}_k$ be the cube with $x \in D_k(x)$. Write

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and define the **magnification operator** $M : \Xi \to \Xi$ by

$$M(x,\mu) = (T_{D_1(x)}(x), \mu^{D_1(x)}).$$

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- A **micromeasure distribution** of μ at x is an accumulation point of the scenery distributions in $\mathcal{P}(\Xi)$ w.r.t. the weak topology.
- The measure component of a micromeasure distribution is supported on the **micromeasures** of μ at x (i.e. accumulation points of the '**minimeasures**' $\mu^{D_k(x)}$, as $k \to \infty$)

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Condition (2) seems strange at first sight, but is essential to carry geometric information from the micromeasure back to μ.

In 'nice' situations, (2) does not cause any problems in the proofs and often $Q_q = Q$ for all $q \in \mathbb{N}$.

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Proposition (Hochman-Shmerkin 2012)

Let μ be a self-similar measure in $\mathbb{R}^d$ satisfying the strong separation condition. Then there exists a Borel-set B with $\mu(B) > 0$ and a similitude S of $\mathbb{R}^d$, such that

$$\nu = \mu(B)^{-1} S(\mu|_B)$$

generates an ergodic CP distribution.

Example: dimensions of projections

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Theorem (Hochman-Shmerkin 2012)

Suppose μ generates an ergodic CP distribution Qand let $k \in \mathbb{N}$ and $\varepsilon > 0$. Then there exists an open dense set $\mathcal{U}_\varepsilon \subset \Pi_{d,k}$ (which is also of full measure) such that for all $\pi \in \mathcal{U}_\varepsilon$

$$\dim_H \pi(\mu) > \min\{k, \dim_H \mu\} - \varepsilon.$$

Example: dimensions of C^1 images

Theorem (Hochman-Shmerkin 2012)

Suppose that a measure μ on $[0,1]^d$ generates an ergodic CP-chain Q. Let $\pi \in \Pi_{d,k}$ and $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for all C^1 maps $g : [0,1]^d \to \mathbb{R}^k$ such that the maximal norm

$$\sup_{x \in supp(\mu)} \|D_x g - \pi\| < \delta,$$

we have

$$\dim_H g\mu > \dim_H \pi\mu - \varepsilon.$$

Tuomas Sahlsten

Tuomas Sahlsten and Andy Ferguson

An application to the distance problem

Theorem (Ferguson, F, Sahlsten, 2013)

If μ *on* $\mathbb{R}^2$ *generates an ergodic CP distribution and* $\mathcal{H}^1(\mathrm{spt}\,\mu) > 0$*, then*

$$\dim_H D(\mathrm{spt}\,\mu) \geq \min\{1, \dim_H \mu\}.$$

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Corollary (Ferguson, F, Sahlsten, 2013)

If E is a Bedford-McMullen carpet with $\dim_H E \geq 1$, then $\dim_H D(E) = 1$.

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• Case 2: $Dir(K)$ is dense in S^1.

An outline of the proof

- Let μ be a probability measure with support $K \subset \mathbb{R}^2$ satisfying $\mathcal{H}^1(K) > 0$ and suppose μ generates an ergodic CP-distribution.

- Define the *direction set* of K by

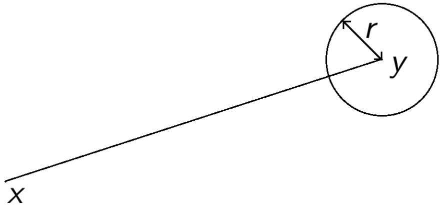
$$Dir(K) = \left\{ \frac{x-y}{|x-y|} : x, y \in K, x \neq y \right\}$$

- Case 1: $Dir(K)$ is not dense in S^1. This means K is 1-rectifiable. Combined with the fact that K has positive length, a result of Besicovitch and Miller gives that $D(K)$ contains an interval.

- Case 2: $Dir(K)$ is dense in S^1. Let $\varepsilon > 0$ and choose $\pi \in Dir(K)$ such that

$$\dim_H \pi(\mu) > \min\{1, \dim_H \mu\} - \varepsilon.$$

An outline of the proof



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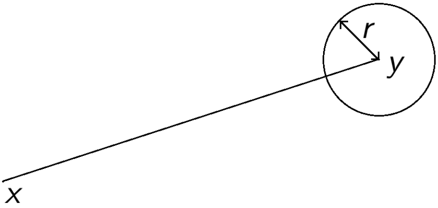

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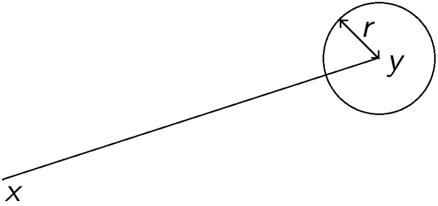

- Let g be the pinned distance map at x, $g(z) = |x - z|$
- Choose $r > 0$ small enough to guarantee that

$$\sup_{z \in B(y,r)} \|D_z g - \pi\| < \delta.$$

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completing the proof.

Conformally generated fractals

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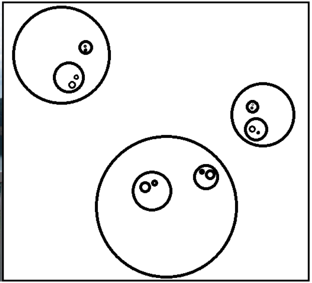

Conformally generated fractals

Proposition (Hochman-Shmerkin 2012)

Let μ be a self-similar measure in $\mathbb{R}^d$ satisfying the strong separation condition. Then there exists a Borel-set B with $\mu(B) > 0$ and a similitude S of $\mathbb{R}^d$, such that

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Proposition (F, Pollicott 2014)

Let μ be a Gibbs measure supported on a self-conformal set satisfying the strong separation condition. Then there exists a Borel-set B with $\mu(B) > 0$, a conformal map S of $\mathbb{R}^d$ and a measure $\mu' \equiv \mu$, such that

$$\nu = \mu'(B)^{-1} S(\mu'|_B)$$

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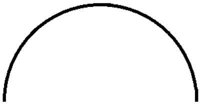

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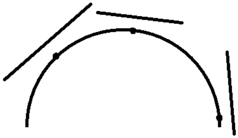

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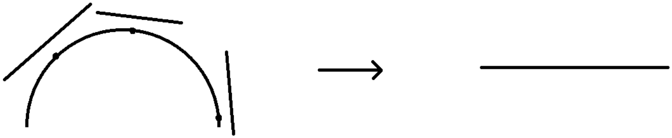

A further application to the distance problem

Theorem (F, Pollicott 2014)

If μ on $\mathbb{R}^2$ generates an ergodic CP distribution and $\mathcal{H}^1(\mathrm{spt}\,\mu) > 0$, then

$$\dim_H D(S(\mathrm{spt}\,\mu)) \geq \min\{1, \dim_H \mu\}$$

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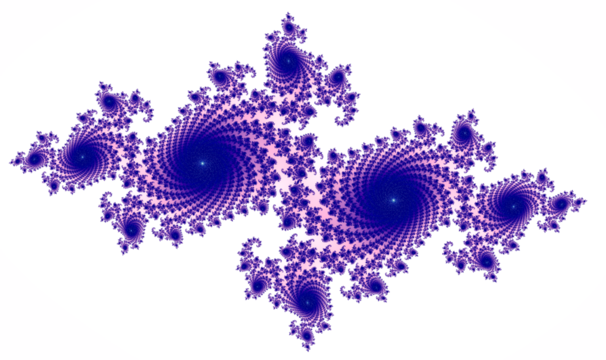
Corollary (F, Pollicott 2014)

If E is a self-conformal set with $\dim_H E > 1$, then $\dim_H D(E) = 1$.



Thank you!

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