Recent progress on the Assouad dimension

Jonathan M. Fraser

The University of Manchester, UK

Joint work with several people!

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$\dim_{\mathrm{H}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F$

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Robinson: *Dimensions, Embeddings, and Attractors*

Heinonen: *Lectures on Analysis on Metric Spaces*.

A motivating example...

Consider the standard Mandelbrot percolation on $[0, 1]^d$ using an M^d grid and probability $p > M^{-d}$.

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We recently learned that the Assouad dimension result follows from earlier work of Berlinkov-Jarvenpää.

Self-similar sets

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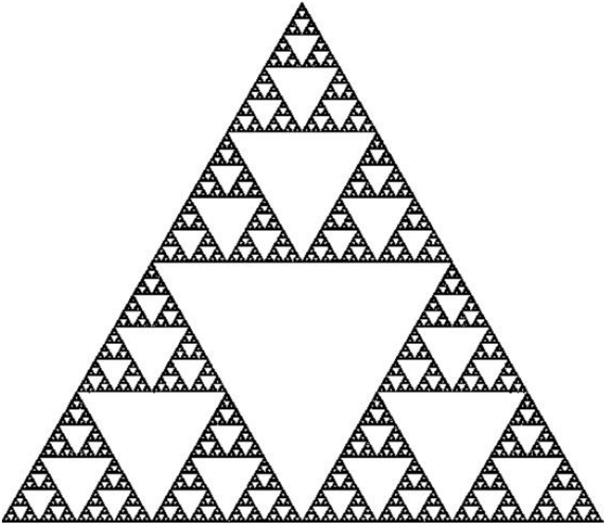
If one can find an open set $\mathcal{O} \subset [0,1]^d$ such that

- $S_i(\mathcal{O}) \subset \mathcal{O}$ for all $i \in \mathcal{I}$
- $S_i(\mathcal{O}) \cap S_j(\mathcal{O}) = \emptyset$ for all $i \neq j \in \mathcal{I}$

then we say the open set condition is satisfied for this IFS.

Self-similar sets



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Proposition (F. '14)

For any $\varepsilon \in (0,1)$, there exists a self-similar set $F \subseteq [0,1]$ with $\dim_H F \leqslant \varepsilon < 1 = \dim_A F$.

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Theorem (F.-Henderson-Olson-Robinson '15)

Let F be a self-similar subset of $[0, 1]$.

- *If the WSP is satisfied, then $\dim_A F \ = \ \dim_H F$.*
- *If the WSP is not satisfied, then $\dim_A F \ = \ 1$.*

Assoud dimension and positive Hausdorff measure

Theorem (Farkas-F. '15)

Let F be a (graph-directed) self-similar subset of $[0,1]^d$ with $\dim_H F = t$.

- *If $\mathcal{H}^t(F) > 0$, then F is Ahlfors regular*

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The proof uses the fact that the t-dimensional Hausdorff content and Hausdorff measure coincide for (graph-directed) self-similar sets.

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Corollary (Farkas-F. '15)

Let F be a self-similar subset of $[0,1]$ with $\dim_H F = t < 1$.

- $\mathcal{H}^t(F) > 0 \Rightarrow \dim_A F = t$.
- $\mathcal{H}^t(F) = 0 \Rightarrow \dim_A F = 1$.

Projections of planar sets

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Let F be an analytic subset of the plane with Hausdorff dimension $s \in [0, 2]$. Then for almost all $\theta \in [0, 2\pi)$

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Note: the almost sure value can be strictly less than $\min\{1, s\}$.

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Theorem (F.-Orponen '15)

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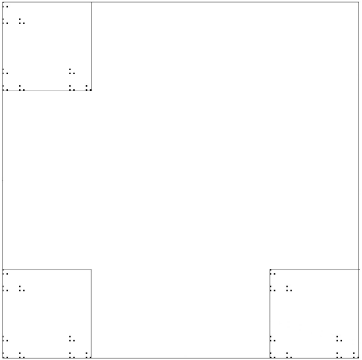
• This is a partial Marstrand Theorem for Assouad dimension.

• We can use self-similar sets to show that a full Marstrand Theorem for Assouad dimension does not exist!

Projections of planar sets

Consider the following example of Peres, Simon and Solomyak from 2000:



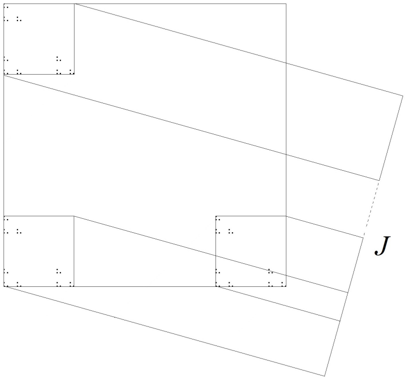
The contraction ratio is $c \in (1/5, 1/3)$, and the Hausdorff dimension is $s = -\log 3 / \log c$.

Projections of planar sets

Theorem (Peres-Simon-Solomyak '00)

There is a non-empty open interval of projections $J \subseteq \{\theta : \pi_\theta \text{ not injective}\}$ such that for almost all $\theta \in J$ we have

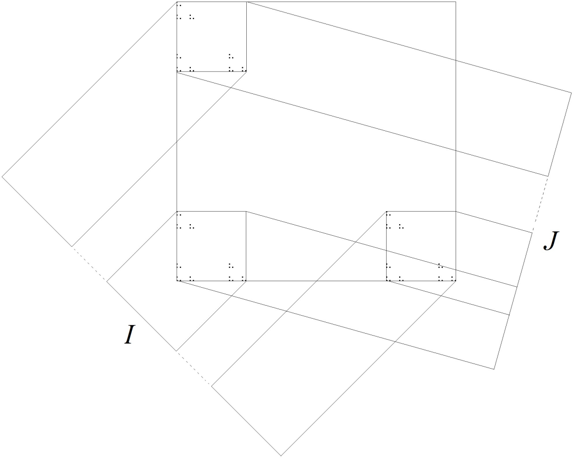
$$\mathcal{H}^s(\pi_\theta F) = 0.$$



Projections of planar sets

Since $c < 1/3$, we can find an open interval I where the projection is self-similar and satisfies the OSC, in particular, for all $\theta \in I$ we have

$$\mathcal{H}^s(\pi_\theta F) > 0.$$



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There are disjoint non-empty intervals $I, J \subseteq [0, 2\pi)$ such that:

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There are disjoint non-empty intervals $I, J \subseteq [0, 2\pi)$ such that:

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- For almost all $\theta \in J$, $\mathcal{H}^s(\pi_\theta F) = 0 \Rightarrow \dim_A \pi_\theta F \ = \ 1$

The Assouad dimension of $\pi_\theta F$ is not almost surely constant!

Projections of self-similar sets

Theorem (F.-Orponen '15)

Let F be a non-trivial planar self-similar set.

If all rotations are rational, then, for a given $\theta \in [0, 2\pi)$, we have:

1. *If* $\mathcal{H}^{\dim_H \pi_\theta F}(\pi_\theta F) > 0$, *then* $\dim_A \pi_\theta F \; = \; \dim_H \pi_\theta F$
2. *If* $\mathcal{H}^{\dim_H \pi_\theta F}(\pi_\theta F) = 0$, *then* $\dim_A \pi_\theta F \; = \; 1$.

If one of the rotations is irrational, then

$$\dim_A \pi_\theta F \; = \; 1$$

for **all** $\theta \in [0, 2\pi)$.

Open questions

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If only two values are possible, are they always $\dim_A F$ *and* 1*?*

Merci de votre attention!

Porquerolles Island, 2011

Some references - all on the ArXiv

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