The mobility of stationary measures

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Overview

We can consider a limit set Λ for a pair of contractions $T_0 : [0,1] \to [0,1]$ and $T_1 : [0,1] \to [0,1]$ on the line.

Question

How does the set Λ change as T_0, T_1 change?

We can consider a stationary probability measure μ for weights p_0, p_1 which is supported on the closed set Λ .

Question

How does the measure μ change as T_0, T_1 and p_0, p_1 change?

- Usually the dependence on T_i is more subtle (interesting?) than the dependence on p_i.
- Generally we need more smoothness in the dependence of p_i and T_i then we can expect from the μ.
- We will make a detour in the exposition to take in the scenery (e.g., the Monge Optimization Problem).

Limits sets of contractions

Let us consider a specific setting of the unit interval $[0, 1]$ and two C^∞ contractions

$$T_0 : [0, 1] \to [0, 1] \text{ and } T_1 : [0, 1] \to [0, 1]$$

Often we will ask (for convenience?) for disjoint images (i.e., $T_0[0, 1] \cap T_1[0, 1] = \emptyset$).

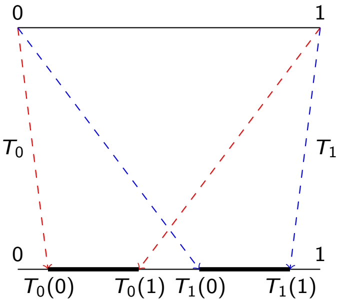


Figure: Two contractions on the unit interval

The limit set

Definition

The limit set $\Lambda = \Lambda(T_0, T_1)$ is the smallest closed set such that $T_0\Lambda \cup T_1\Lambda = \Lambda$

Equivalently, and perhaps more intuitively, we can define the set by

$$\Lambda = \left\{ \lim_{n\to\infty} T_{i_0} \cdots T_{i_n}(0) \,:\, i_0, \cdots, i_n \in \{0,1\} \right\}$$

1. The set Λ is a Cantor set (when $T_0[0,1] \cap T_1[0,1] = \emptyset$).
2. The construction of Λ is often called an “iterated function scheme” or “cookie cutter” (but as seldom as possible by me).

When you get bored try counting the number of mistakes in these slides. First to spot 100 should shout “Bingo”

Example: middle third Cantor set

We can let

$$T_0(x) = \frac{x}{3} \text{ and } T_1(x) = \frac{x}{3} + \frac{2}{3}$$

then Λ is the usual middle third Cantor set, i.e.,

$$\Lambda = \left\{ \sum_{n=0}^{\infty} \frac{i_n}{3^{n+1}} \, : \, i_0, i_1, i_2, i_3, \cdots \in \{0, 2\} \right\}.$$

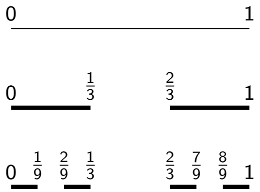


Figure: The usual construction of the middle third Cantor set

More examples

1. More generally, for any $0 < \lambda < \frac{1}{2}$ we can let

$$T_0(x) = \lambda x \text{ and } T_1(x) = \lambda x + (1 - \lambda)$$

then Λ is the usual middle $(1 - 2\lambda)$-Cantor set.

(This is a "self-similar" set, since the contractions for T_0, T_1 are the same).

2. Even more generally, for any $0 < \lambda_1, \lambda_2 < 1$ with $\lambda_1 + \lambda_2 = 1$ we can let

$$T_0(x) = \lambda_1 x \text{ and } T_1(x) = \lambda_2 x + (1 - \lambda_2)$$

and again Λ is a limit set.

(The contractions for T_0, T_1 may now be different making the Cantor set seem lopsided).

3. For a nonlinear example we can consider

$$T_0(x) = \frac{1}{x+2} \text{ and } T_1(x) = \frac{1}{x+7}$$

then the limit set Λ consist of those $0 < x < 1$ whose continued fraction expansions consist only of digits 2 and 7.

Stationary measures

We next want to introduce probability measures supported on the limit set Λ.

The choice of measure will be determined by C^∞ weight functions $p_0, p_1 : [0,1] \to (0,1)$ such that $p_0(x) + p_1(x) = 1$.

Definition

We say that a probability μ is a stationary measure if for any $f \in C^0([0,1], \mathbb{R})$ we have that

$$\int f(x) d\mu(x) = \int \left(p_0(x) f(T_0 x) + p_1(x) f(T_1 x)\right) d\mu(x).$$

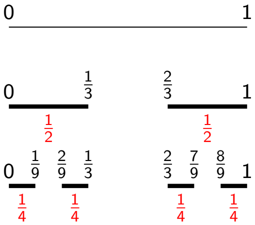
Equivalently, we can also construct μ by:

$$\int f d\mu = \lim_{n \to +\infty} \sum_{i_0, \cdots, i_{n-1} \in \{0,1\}} p_{i_0}(0) p_{i_1}(T_{i_0} 0) \cdots p_{i_{n-1}}(T_{i_0} \cdots T_{i_{n-2}} 0) f(T_{i_0} \cdots T_{i_{n-1}} 0)$$

(i.e., as a limit of suitably weighted measures on 2^n image points).

$(1/2, 1/2)$-Bernoulli measures on middle third Cantor set

If we let $T_0 = \frac{x}{3}$, $T_1(x) = \frac{x}{3} + \frac{1}{3}$ and $p_0 = p_1 = \frac{1}{2}$ then the measure is the natural $(1/2, 1/2)$-Bernoulli measure.



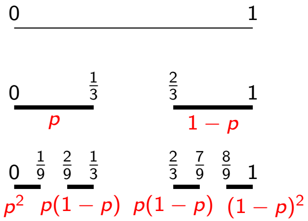
In particular, $\mu\left[0, \frac{1}{3}\right] = \mu\left[\frac{2}{3}, 1\right] = \frac{1}{2}$,

$$\mu\left[0, \frac{1}{9}\right] = \mu\left[\frac{2}{9}, \frac{1}{3}\right] = \mu\left[\frac{2}{3}, \frac{7}{9}\right] = \mu\left[\frac{8}{9}, 1\right] = \frac{1}{4},$$

and in general, $\mu\left[\sum_{n=0}^{N-1} \frac{i_n}{3^{n+1}}, \sum_{n=1}^{N} \frac{i_n}{3^{n+1}} + \frac{1}{2^N}\right] = \frac{1}{3^N}$ for $i_0, \cdots, i_{N-1} \in \{0, 2\}$.

$(p, 1-p)$-Bernoulli measures

If we let $p_0 = p$ and $p_1 = 1 - p$ then the measure is the natural $(p, 1-p)$-Bernoulli measure.



$$\text{Thus } \mu\left[0, \frac{1}{3}\right] = p, \mu\left[\frac{2}{3}, 1\right] = 1 - p$$

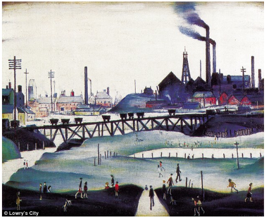
$$\mu\left[0, \frac{1}{9}\right] = p^2, \mu\left[\frac{2}{9}, \frac{1}{3}\right] = p(1-p), \mu\left[\frac{2}{3}, \frac{7}{9}\right] = p(1-p), \mu\left[\frac{8}{9}, 1\right] = (1-p)^2$$

$$\text{and} \mu\left[\sum_{n=0}^{N-1} \frac{i_n}{3^{n+1}}, \sum_{n=0}^{N-1} \frac{i_n}{3^{n+1}} + \frac{1}{3^N}\right] = p^N \prod_{n=0}^{N-1} \left(\frac{1}{p} - 1\right)^{i_n/2} \text{ for } i_0, \cdots, i_{N-1} \in \{0, 2\}.$$

Monge Transportation problem (1781)

Assume we have a number of mines producing iron ore and a number of factories which need to be supplied.Assume that the mine at $x \in [0,1]^d$ supplies the factory at $y \in [0,1]^d$ and that the cost of transporting the ore from mine to factory is proportional to the distance $|x-y|$.

The problem is to minimise the total cost over different choices of pairings x to y.



We can approximate the distribution of mines by a probability μ and the distribution of factories by a probability ν. When possible, we want to find a map $T : [0,1]^d \to [0,1]^d$ (i.e., $T(x) = y$) which minimises

$$\inf \left\{ \int_X |x - T(x)| d\mu(x) : T_*\mu = \nu \right\}.$$

Example (Trivial example, $d = 1$)

If μ, ν have no atoms and are supported on $[0,1]$ there is a solution $T = F_\nu^{-1} \circ F_\mu : [0,1] \to [0,1]$ where $F_\mu(x) = \mu([0,x])$ and $F_\nu(x) = \nu([0,x])$ for $0 < x < 1$.

Gaspard Monge (1746-1818)

Gaspard Monge had a remarkably successful career under three rather different types of government in France.

- A prodigy, he was made a professor at the age of 22 at the École Royale du Génie at Mézières in pre-revolutionary France. He invented descriptive geometry (representing three dimensional figures in two dimensions, for example) but the theory was surpressed as a military secret for many years.
- The son of a wine merchant, he was a keen supporter of the French Revolution and was Minister for the Navy. In this post he was kind to Napoleon, then a young officer. During the Terror he was denounced, but escaped execution.
- Subsequently, he became one of Napoleon's closest friends during the Emperor's reign, and Director of the École Polytechnique.

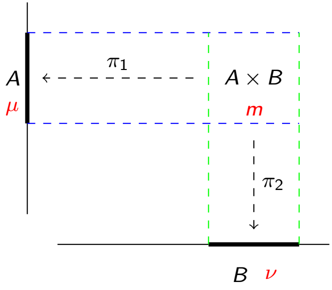
His name is one of the 72 names of scientists inscribed on the Eiffel tower

The optimisation problem

A more general formulation due to Kantorovich is to minimise

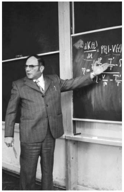
$$d(\mu,\nu) := \inf\left\{\int_{X\times X} |x-y|dm(x,y) \,:\, \pi_1 m = \mu, \pi_2 m = \nu\right\}$$

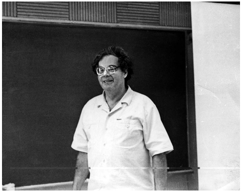
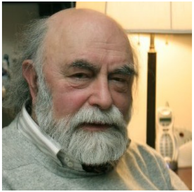
where the infimum is over probability measures m on $X \times X$ projecting to ν and μ.



i.e., $\pi_1 m(A) = m(A \times X)$ and $\pi_2 m(B) = m(X \times B)$.

Kantorovich-Wasserstein metric



Kantorovich's work (from 1942) lead to the Nobel Prize in Economics in 1975.

An equivalent definition using Lipschitz functions is in a paper of Kantorovich and Rubinshtein from 1958.

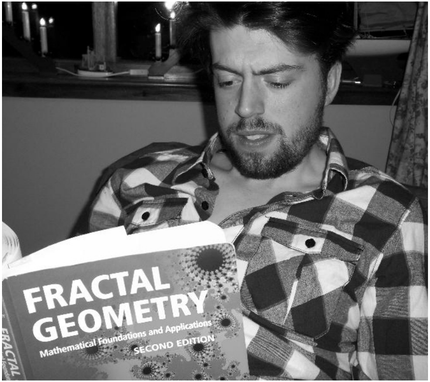
$$d(\mu,\nu) := \sup\left\{ \left| \int f d\mu - \int f d\nu \right| \ : \ \|f\|_{Lip} \leq 1 \right\} \text{ where } \|f\|_{Lip} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

The name "Wasserstein distance" was coined by Dobrushin in 1970, after the Russian mathematician Leonid Wasserstein who re-discovered the concept in 1969. A vigorous defence of Kantorovich's contribution appears in a 2005 article of his friend Vershik.

I once collected Dobrushin (who was a big mathematician, in all senses of the word) from Coventy train station in a Ford Fiesta (which was a smallish car). The same year I also collected Vershik from the same station in the same car.

Jon's paper on the Wasserstein-Kantorovich metric

While at Warwick, Jon wrote a short paper which computed explicitly the Wasserstein-Kantorovich metric for very special examples of stationary measures.



First and second moments for self-similar couplings and Wasserstein distances

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Abstract

We study aspects of the Wasserstein distance in the context of self-similar measures. Computing this distance between two measures involves minimising certain moment integrals over the space of *couplings*, which are measures on the product space with the original measures as prescribed marginals. We focus our attention on self-similar measures associated to equicontractive iterated function systems satisfying the open set condition and consisting of two maps on the unit interval. We are particularly interested in understanding the restricted family of *self-similar* couplings and our main achievement is the explicit computation of the 1st and 2nd moment integrals for such couplings. We show that this family is enough to yield an explicit formula for the 1st Wasserstein distance and provide non-trivial upper and lower bounds for the 2nd Wasserstein distance.

Mathematics Subject Classification 2010: Primary: 28A80, 28A33, 60B05. Secondary: 28A78.

Key words and phrases: Wasserstein metric, self-similar measure, self-similar coupling.

1 Introduction

The Wasserstein metric is widely used as an informative and computable distance function between mass distributions. In computer science it is commonly referred to as the 'earth mover's distance' and is a measure of the 'work' required to change one distribution into the other. For discrete distributions on finite sets one can develop efficient algorithms to determine the distance, but in the non-discrete setting calculations can be far from trivial and involve minimising certain moment integrals over the space of *couplings*, which are measures on the product space with the original measures as prescribed marginals.

In this paper we study the 1st and 2nd moment integrals for self-similar couplings of pairs of self-similar measures arising from equicontractive iterated function systems satisfying the open set condition (OSC) and consisting of two maps on the unit interval. Given two such measures, the family of self-similar couplings is a 1-parameter family and we are able to give an explicit formula for the 1st and 2nd moments for all measures in this family in terms of this parameter and the defining parameters of the original measures. This gives natural upper bounds on the 1st and 2nd Wasserstein distances between the original measures and leads us to the following natural questions: 'Can the Wasserstein distances be realised by self-similar couplings?' and 'how do the 1st and 2nd moment integrals depend on the defining parameters?' In the case of the 1st distance, we use the Kantorovich-Rubinstein duality theorem, which involves maximising the integral of 1-Lipschitz test functions with respect to the difference of the two measures, to prove that self-similar couplings are indeed sufficient. We thus derive an explicit formula for the 1st Wasserstein distance in terms of the different probability vectors, the contraction parameter, and the translation vectors and, moreover, can exhibit an explicit coupling which realises the distance. Once we have the formula for the 1st Wasserstein distance we are able to make the following peculiar observation. If the translation vectors are chosen such that the end points of the unit interval are in the support of the measure (i.e. the support is the middle $(1-2c)$ Cantor set), then the 1st Wasserstein

1

Figure: (a) Jon relaxing with a good book; (b) Another good read.

Jon's Example

We can consider the Lipschitz function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \lambda x$ with $-1 \leq \lambda \leq 1$. One sees that $\|f\|_{Lip} = |\lambda| \leq 1$.

Let us consider two stationary measures:

1. $T_0(x) = cx + t_0$ with $(p, 1-p)$-Bernoulli stationary measure μ_{p,t_0}, and
2. $T_1(x) = cx + t_1$ with $(q, 1-q)$-Bernoulli stationary measure μ_{q,t_1}.

Theorem (J. Fraser, Proposition 2.5)

$$\int f(x) d\mu_{p,t_0}(x) - \int f(x) d\mu_{q,t_1}(x) = \frac{\lambda(p-q)(t_1-t_2)}{1-c}$$

In particular, we immediately see that

$$(p,t) \mapsto \int f(x) d\mu_{p,t}(x)$$

is real analytic.

So at least in this nice and explicit case we see that the dependence of the integrals of stationary measures has an analytic dependence.

Jon's Example

In fact in this example it is possible to explicitly compute the Kantorovich-Wasserstein distance between the two stationary measures.

We recall that

$$d(\mu_{p,t_0},\mu_{q,t_1}) = \sup\left\{\left|\int f(x)d\mu_{p,t_0}(x) - \int f(x)d\mu_{q,t_1}(x)\right| \,:\, \|f\|_{Lip}\le 1\right\}.$$

Theorem (J. Fraser, Corollary 2.6)

One can show

$$d(\mu_{p,t_0},\mu_{q,t_1}) = \frac{\lambda|p-q|(t_1-t_2)}{1-c}.$$

Moreover, the supremum in the definition of the metric is realised by $f(x)=x$ or $f(x)=-x$.

Question: How does the measure change in general?

We can consider the regularity of the dependence of the measures in the general setting. Assume that $T_0, T_1 : [0,1] \to [0,1]$ and $p_0, p_1 : [0,1] \to [0,1]$ are all C^k.

Assume first that we make a C^k perturbation in the weights.

Question

How does the set μ change as p_0, p_1 change? More precisely, if $f : [0,1] \to \mathbb{R}$ is C^∞ then what is the dependence of $\int f d\mu$?

Assume next that we make a C^k perturbation in the contractions.

Question

How does the measure μ change as T_0, T_1 change? More precisely, if $f : [0,1] \to \mathbb{R}$ is C^∞ then what is the dependence of $\int f d\mu$?

Varying the weights (linear example)

Let us consider

- $T_0(x) = \frac{1}{4}x$ and $T_1(x) = \frac{1}{2}x + \frac{1}{2}$ and
- weights $(p, 1-p)$ where $0 < p < 1$.

with stationary measure μ.

Let $f(x) = \sin(2\pi x)$ and we can consider $p \mapsto \int f d\mu$.

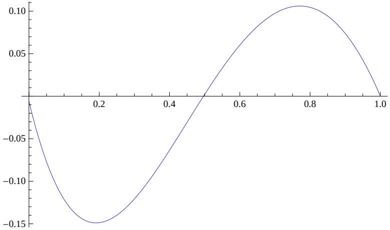


Figure: A plot of $\int f d\mu$ against $0 < p < 1$

Varying the contractions (linear example)

Let us consider

- $T_0(x) = cx$ (with $0 < c < \frac{1}{2}$) and $T_1(x) = \frac{1}{2}x + \frac{1}{2}$ and
- weights $(\frac{1}{2}, \frac{1}{2})$.

with stationary measure μ. Let $f(x) = \sin(2\pi x)$ and we can consider $p \mapsto \int f d\mu$.

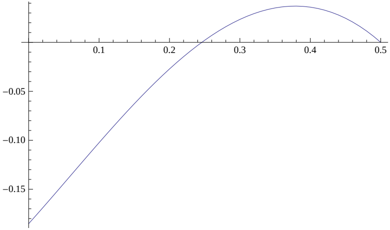


Figure: A plot of $\int f d\mu$ against $0 < c < \frac{1}{2}$

Varying the weights (non-linear example)

Let us consider

- $T_0(x) = \frac{1}{x+2}$ and $T_1(x) = \frac{1}{x+7}$ and
- weights $(p, 1-p)$ where $0 < p < 1$.

with stationary measure μ.

Let $f(x) = \sin(2\pi x)$ and we can consider $p \mapsto \int f d\mu$.

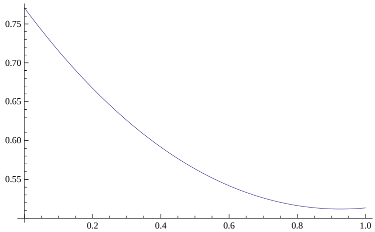


Figure: A plot of $\int f d\mu$ against $0 < p < 1$

Varying the contractions (non-linear example)

Let us consider

- $T_0(x) = \frac{1}{x+2}$ and $T_1(x) = \frac{1}{x+c}$ (with $3 < c < 10$) and
- weights $(\frac{1}{2}, \frac{1}{2})$.

with stationary measure μ.

Let $f(x) = \sin(2\pi x)$ and we can consider $p \mapsto \int f d\mu$.

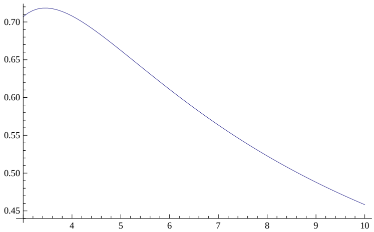


Figure: A plot of $\int f d\mu$ against $3 < c < 10$

Some results : Italo Cipriano et moi

We can consider the case of C^k weights and contractions (with $k \geq 3$).

Theorem (Change of Weights)

Consider C^k perturbations

$$(\epsilon, \epsilon) \ni \lambda \mapsto p_0^{(\lambda)}, p_1^{(\lambda)} \in C^k([0,1], \mathbb{R}).$$

Then for any $f \in C^\infty([0,1], \mathbb{R})$ we have that $(-\epsilon, \epsilon) \ni \lambda \mapsto \int f d\mu_\lambda \in \mathbb{R}$ is C^k.

Here we can view $C^k([0,1], \mathbb{R})$ as a Banach space and "C^k perturbation" means that for any $L \in C^k([0,1], \mathbb{R})^*$ the composition $(-\epsilon, \epsilon) \ni \lambda \mapsto L(p_i^{(\lambda)}) \in \mathbb{R}$ is C^k.

Theorem (Change of contractions)

Consider C^k perturbations

$$(-\epsilon, \epsilon) \ni \lambda \mapsto T_0^{(\lambda)}, T_1^{(\lambda)} \in C^k([0,1], [0,1]).$$

Then for any $f \in C^\infty([0,1], \mathbb{R})$ we have that $(-\epsilon, \epsilon) \ni \lambda \mapsto \int f d\mu_\lambda \in \mathbb{R}$ is C^{k-2}.

Here we can view $C^k([0,1], [0,1]) \subset C^k([0,1], \mathbb{R})$ as a Banach manifold - which locally is modelled by a Banach space.

Italo Cipriano et moi

Figure: (a) Italo Cipriano on Monday, sitting beneath “DNA Quilt”; (b) M.P. on Thursday, standing in front of a Menger sponge.

Some of the proof

The proof follows a natural course (if you are perverse enough to use a thermodynamic approach).

1. We can use symbolic dynamics to code the limit set by a sequence space $\Sigma = \{0,1\}^{\mathbb{Z}^+}$ using the (α-Hölder) map:

$$\pi^{(\lambda)} : \Sigma \to [0,1]$$
$$\pi^{(\lambda)}\left((x_n)_{n=0}^{\infty}\right) = \lim_{n\to+\infty} T_{x_0}^{(\lambda)} T_{x_1}^{(\lambda)} \cdots T_{x_n}^{(\lambda)}(0)$$

2. Given $f : [0,1] \to \mathbb{R}$ we can rewrite

$$\int_{[0,1]} f d\mu_\lambda = \int_\Sigma f \circ \pi^{(\lambda)} d\nu_\lambda$$

   where ν_λ is the Gibbs measure associated to $\log |T'_{x_0}(\pi^{(\lambda)}(x_n))|$.

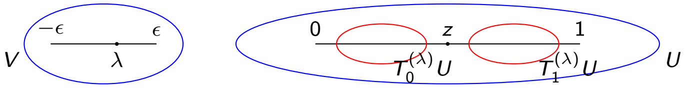
3. We can show smoothness of the map $\lambda \mapsto \pi^{(\lambda)} \in C^\alpha(\Sigma, \mathbb{R})$ and deduce smoothness of $\lambda \mapsto \log |T'_{x_0}(\pi^{(\lambda)}((x_n)_{n=0}^{\infty}))|$ and $\lambda \mapsto \nu_\lambda$.
   (The loss of two derivatives comes from properties of the "composition" map).

A particularly simple setting: Analytic functions

Let $p_0^{(\lambda)}, p_1^{(\lambda)} : [0,1] \to (0,1)$ and $T_0^{(\lambda)}, T_1^{(\lambda)} : [0,1] \to [0,1]$ be C^ω for $\lambda \in (-\epsilon, \epsilon)$, i.e., there are neighbourhoods: $[0,1] \subset U \subset \mathbb{C}$; and $(-\epsilon, \epsilon) \subset V \subset \mathbb{C}$, such that

$$U \times V \ni (z, \lambda) \mapsto p_0^{(\lambda)}(z), p_1^{(\lambda)}(z) \in \mathbb{C} \text{ and}$$
$$U \times V \ni (z, \lambda) \mapsto T_0^{(\lambda)}(z), T_1^{(\lambda)}(z) \in \mathbb{C}$$

are analytic.



Theorem

For any C^ω function $f : [0,1] \to \mathbb{R}$ we have that

$$(-\epsilon, \epsilon) \ni \lambda \mapsto \int f d\mu_\lambda$$

is C^ω, i.e., there is an open neighbourhood $(-\epsilon, \epsilon) \subset V' \subset \mathbb{C}$ such that $V' \ni \lambda \mapsto \int f d\mu_\lambda \in \mathbb{C}$ is analytic.

Most of one proof: Ingredients

We can formally define a function ("zeta function")

$$d(z,\lambda,u) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{|\underline{i}|=n}(T_{\underline{i}}^{(\lambda)})'(x_{\underline{i}}^{(\lambda)})\exp\left(uf^n(x_{\underline{i}}^{(\lambda)})\right)\right)$$

where:

- $T_{\underline{i}}^{(\lambda)} := T_{i_0}^{(\lambda)} \circ \cdots \circ T_{i_{n-1}}^{(\lambda)}$ for $\underline{i} = (i_0, \cdots, i_{n-1})$ and $|\underline{i}| = n$;
- $T_{\underline{i}}^{(\lambda)}(x_{\underline{i}}^{(\lambda)}) = x_{\underline{i}}^{(\lambda)}$ is its fixed point; and
- $f^n(x_{\underline{i}}^{(\lambda)}) := \sum_{k=0}^{n-1} f(x_{\sigma^k\underline{i}}^{(\lambda)})$ where $\sigma^k\underline{i} = (i_k, i_{k+1}\cdots, i_{n-1}, i_0, \cdots, i_{k-1})$.

Lemma (Standard stuff)

1. *$d(z,\lambda,u)$ is analytic in each variable for $|z|$ sufficiently small.*
2. *Each $(z,u)\mapsto d(z,\lambda,u)$ has an analytic extension to a neighbourhood of $(1,0)$.*
3. $\int f d\mu_\lambda = \frac{\partial d(1,\lambda,u)}{\partial u}|_{u=0} / \frac{\partial d(z,\lambda,0)}{\partial z}|_{z=1}$.

Thus to prove analyticity of $\lambda \mapsto \int f d\mu_\lambda$ it suffices to show analyticity of $d(z,\lambda,u)$ in a neighbourhood of $(1,0,0)$.

Most of the proof: Analyticity of $d(z, \lambda, u)$

However, we can establish this analyticity of $d(z, \lambda, u)$ by:

1. showing analyticity of the contributions from each of the fixed points;
2. bundling the individual analyticity together in the complex function.

More precisely,

- By the implicit function theorem, for each string $\underline{i}$ there exists a neighbourhood $(-\epsilon, \epsilon) \subset V_{\underline{i}} \subset \mathbb{C}$ such that the fixed point

$$V_{\underline{i}} \ni \lambda \mapsto x_{\underline{i}}^{(\lambda)}$$

  is analytic on a complex neighbourhood $V_{\underline{i}} \supset (-\epsilon, \epsilon)$.
- The intersection $V' := \cap_{\underline{i}} V_{\underline{i}} \subset \mathbb{C}$, is still a neighbourhood of $(-\epsilon, \epsilon)$ in $\mathbb{C}$. (This is an exercise using the fact that the $T_{\underline{i}}^{(\lambda)} : [0,1] \to [0,1]$ are contracting).

Thus we can deduce that $d(z, \lambda, u)$ is analytic in a neighbourhood of $(1, 0, 0)$, as required.

Alternatively, we could use a transfer operator and perturbation theory approach - but the above avoids infinite dimensional spaces

Final question - just so that I don't end on a proof :(

Assume that we are given for $\lambda \in [a, b]$:

- a C^∞ family of contractions $T_0^{(\lambda)}, \cdots, T_{k-1}^{(\lambda)} : [0,1] \to [0,1]$; and
- a C^∞ family of probability weights $p_0^{(\lambda)}, \cdots, p_{k-1}^{(\lambda)} : [0,1] \to (0,1)$.

Let μ_λ be the associated stationary measure, i.e., $\sum_{i=0}^{k-1} p_i^{(\lambda)}(T_i^{(\lambda)}\mu_\lambda) = \mu_\lambda$.

Question

How regular is the function

$$[a, b] \ni \lambda \mapsto d(\mu^{(\lambda)}, \mu^{(a)}) := \sup\left\{ \left| \int f d\mu^{(\lambda)} - \int f d\mu^{(a)} \right| : \|f\|_{Lip} \leq 1 \right\}?$$

Which functions maximise the supremum?

Final slide

Good luck to Jon and congratulations

to Manchester on appointing him.