Interpolating between dimensions

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Joint work with several people

Fractal Geometry and Stochastics VI

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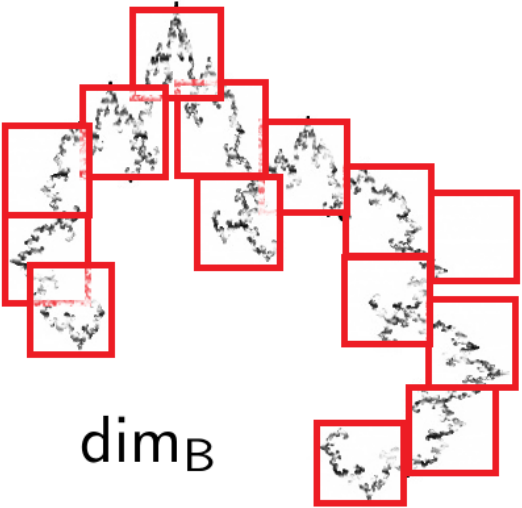

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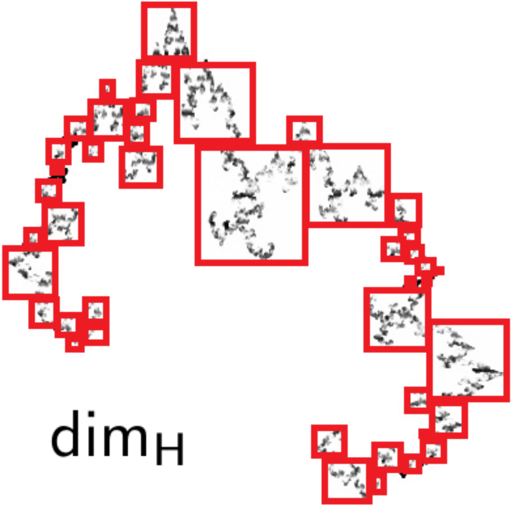

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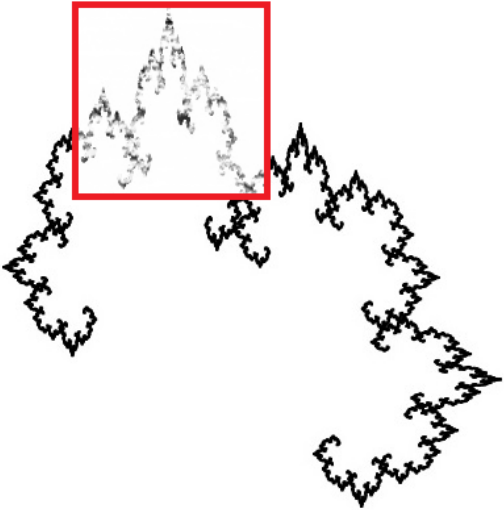

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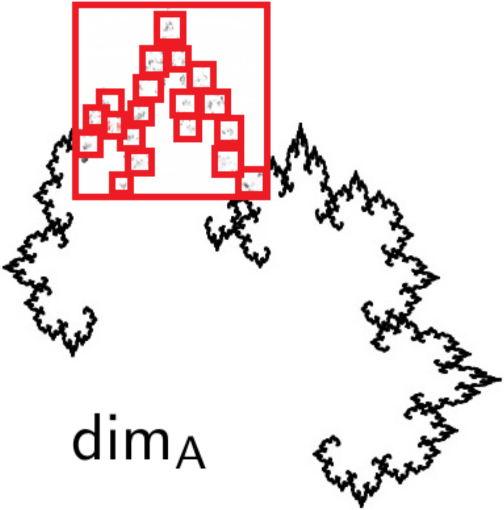

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- If F is Ahlfors regular then $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \dim_{\mathrm{A}} F$.

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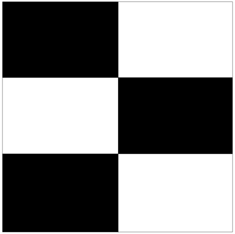
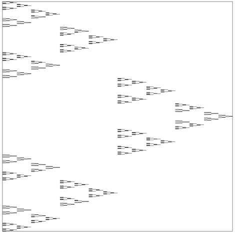
It should be clear that $[0,1]$ is a microset (zoom in at 0).

Examples - self-affine sets

Divide $[0, 1]^2$ into an $m \times n$ grid, where $n > m$ and select a collection of N subrectangles across N_0 columns, with N_i in ith column





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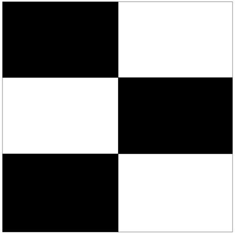
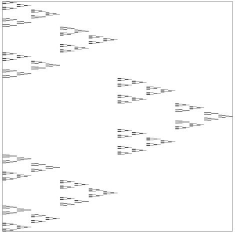




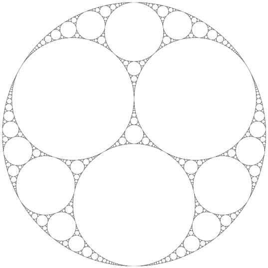
$$\dim_{\mathrm{H}} F \;=\; \frac{\log \sum_i N_i^{\log m/\log n}}{\log m} \qquad \text{(Bedford-McMullen 1985)}$$

$$\dim_{\mathrm{B}} F \;=\; \frac{\log N_0}{\log m} + \frac{\log(N/N_0)}{\log n} \qquad \text{(Bedford-McMullen 1985)}$$

$$\dim_{\mathrm{A}} F \;=\; \frac{\log N_0}{\log m} + \max_i \frac{\log N_i}{\log n} \qquad \text{(Mackay 2011)}$$

Examples - Kleinian limit sets

Let Γ be a geometrically finite Kleinian group acting on d-dimensional hyperbolic space with limit set F. Write $\delta(\Gamma)$ for the Poincaré exponent and $k(\Gamma)$ for the maximal rank of a free Abelian group in the stabiliser of a parabolic fixed point.



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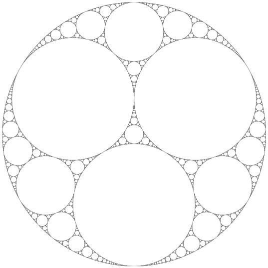

$$\dim_{\mathrm{H}} F = \delta(\Gamma) \qquad \text{(Patterson 1976, Sullivan 1984)}$$

$$\dim_{\mathrm{B}} F = \delta(\Gamma) \qquad \text{(Stratmann-Urbański 1996, Bishop-Jones 1997)}$$

$$\dim_{\mathrm{A}} F = \max\{\delta(\Gamma), k(\Gamma)\} \qquad \text{(F 2017)}$$

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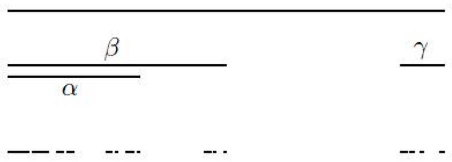
Let $F \subset \mathbb{R}$ be a (non-trivial) self-similar set.



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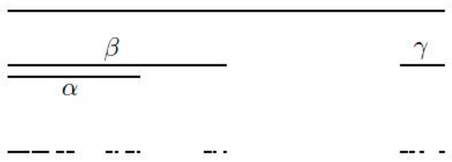

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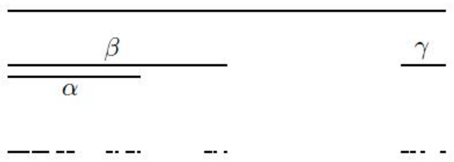

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If WSC is satisfied:

$$\dim_{\mathrm{A}} F \;=\; \dim_{\mathrm{H}} F \;=\; \dim_{\mathrm{B}} F \qquad \text{(F-Henderson-Olson-Robinson 2015)}$$

If WSC fails (e.g., if $\log \alpha / \log \beta \notin \mathbb{Q}$ above):

$$\dim_{\mathrm{A}} F \;=\; 1 \qquad \text{(F-Henderson-Olson-Robinson 2015)}$$

Towards interpolation

Given dimensions dim and Dim which generally satisfy $\dim F \leqslant \operatorname{Dim} F$ we wish to understand the gap between the dimensions by introducing an interpolation function $d : [0, 1] \to \mathbb{R}^+$ which (ideally) satisfies:

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- good fun

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Recall

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F-Hare-Hare-Troscheit-Yu 2018: $\dim_A^\theta F \to \dim_{qA} F$ as $\theta \to 1$.

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Falconer-F-Kempton 2018: Given $\theta \in (0,1)$, we restrict the range of available covers by insisting that $|U_i| \leqslant |U_j|^\theta$ for all i, j

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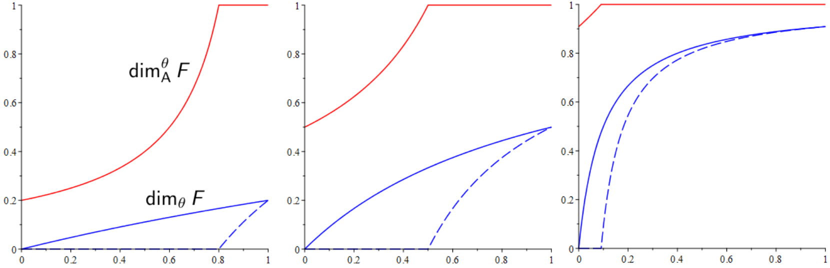

$p = 4$ $\qquad$ $p = 1$ $\qquad$ $p = 1/10$

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F-Yu 2017: For $\theta \in (0, \log m / \log n]$

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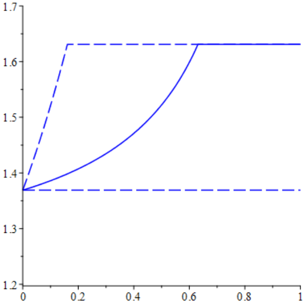

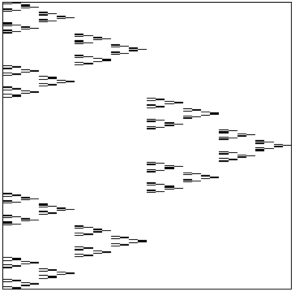

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