Self-similar sets: Projections, Sections and Percolation

Kenneth Falconer

University of St Andrews, Scotland, UK

Summary

- Motivation - projection theorems
- Self-similar sets
- Projections of self-similar sets
- Fractal percolation
- Projections of percolation sets
- Sections or slices of sets
- Projections $\rightarrow$ fractal percolation $\rightarrow$ sections

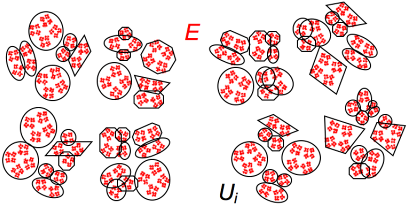
Joint work with Xiong Jin (Manchester)

Hausdorff dimension

Throughout this talk we will generally work in $\mathbb{R}^2$.

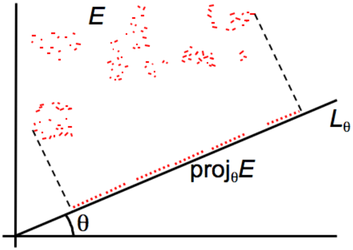
The Hausdorff dimension of a set $E \subset \mathbb{R}^2$ is

$$\dim_H E = \inf \Big\{ s : \text{for all } \epsilon > 0 \text{ there is a countable cover } \{U_i\} \text{ of } E \text{ such that } \sum_i (\text{diam } U_i)^s < \epsilon \Big\}.$$



The Hausdorff dimension of a positive finite Borel measure μ on $\mathbb{R}^n$ is

$$\dim_H \mu = \inf \Big\{ \dim_H K : \mu(K) > 0 \Big\}.$$



Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be a Borel set.
For all $\theta \in [0, \pi)$

(i) $\dim_H \text{proj}_\theta E \leq \min\{\dim_H E, 1\}$.

For almost all $\theta \in [0, \pi)$,

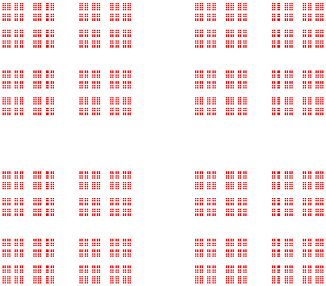
(ii) $\dim_H \text{proj}_\theta E = \min\{\dim_H E, 1\}$,

(iii) $\mathcal{L}(\text{proj}_\theta E) > 0$ if $\dim_H E > 1$.

[proj_θ denotes orthogonal projection onto the line L_θ, $\dim_H$ is Hausdorff dimension, $\mathcal{L}$ is Lebsegue measure on L_θ.]

Exceptional directions

Marstrand's theorem tells nothing about which particular directions may have projections with dimension or measure smaller than normal, i.e. when $\dim_H \mathrm{proj}_\theta E < \min\{\dim_H E, 1\}$, or $\dim_H E > 1$ and $\mathcal{L}(\mathrm{proj}_\theta E) = 0$.



The set shown has dimension $\log 4/\log(5/2) = 1.51$, but with some projections of dimension < 1.

Exceptional directions

The set of exceptional directions can't be 'too big':

Theorem (Kaufman, 1968) If $E \subseteq \mathbb{R}^2$ and $\dim_H E \leq 1$,

$$\dim_H \{\theta : \dim_H \mathrm{proj}_\theta E < \dim_H E\} \leq \dim_H E.$$

– follows from an energy estimate

Theorem (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_H E > 1$,

$$\dim_H \{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_H E.$$

– proof uses Fourier transforms.

Exceptional directions

The set of exceptional directions can't be 'too big':

Theorem (Kaufman, 1968) If $E \subseteq \mathbb{R}^2$ and $\dim_H E \leq 1$,

$$\dim_H\{\theta : \dim_H \mathrm{proj}_\theta E < \dim_H E\} \leq \dim_H E.$$

– follows from an energy estimate

Theorem (F, 1982) If $E \subseteq \mathbb{R}^2$ and $\dim_H E > 1$,

$$\dim_H\{\theta : \mathcal{L}(\mathrm{proj}_\theta E) = 0\} \leq 2 - \dim_H E.$$

– proof uses Fourier transforms.

General problem: find sets or measures or classes of sets where there are no exceptional directions for projections or where the exceptional directions can be identified.

Here we consider self-similar sets and their random counterparts.

Self-similar sets

Given an iterated function system of contracting similarities $f_1, \ldots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ there exists a unique non-empty compact $E \subset \mathbb{R}^2$ called a self-similar set such that

$$E = \bigcup_{i=1}^{m} f_i(E). \qquad (*)$$

We assume throughout that the union $(*)$ is disjoint or perhaps 'nearly disjoint' (i.e. OSC).
Write the similarities as

$$f_i(x) = r_i O_i(x) + t_i$$

where $0 < r_i < 1$ is the scale factor, O_i is a rotation and t_i is a translation.

Given an iterated function system of contracting similarities $f_1, \ldots, f_m : \mathbb{R}^2 \to \mathbb{R}^2$ there exists a unique non-empty compact $E \subset \mathbb{R}^2$ called a self-similar set such that

$$E = \bigcup_{i=1}^{m} f_i(E). \qquad (*)$$

We assume throughout that the union $(*)$ is disjoint or perhaps 'nearly disjoint' (i.e. OSC).
Write the similarities as

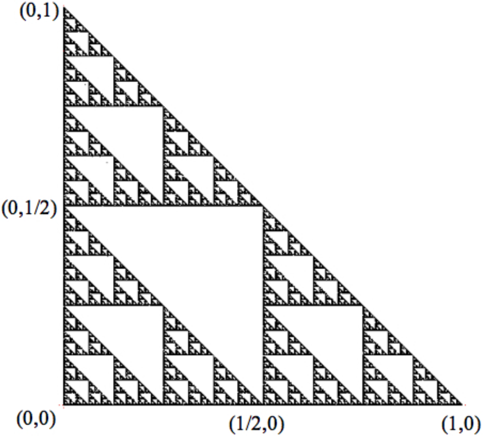
$$f_i(x) = r_i O_i(x) + t_i$$

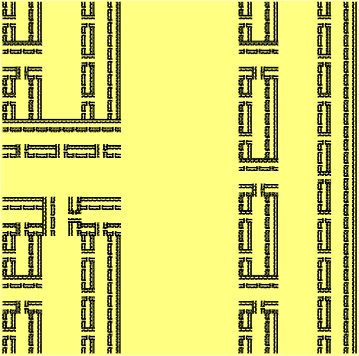
where $0 < r_i < 1$ is the scale factor, O_i is a rotation and t_i is a translation.
The family $\{f_1, \ldots, f_m\}$ has dense rotations if at least one of the O_i is a rotation by an irrational multiple of π, equivalently if group$\{O_1, \ldots, O_m\}$ is dense in $SO(2, \mathbb{R})$.
Otherwise $\{f_1, \ldots, f_m\}$ has finite rotations.

Example: (Right-angled) Sierpiński triangle

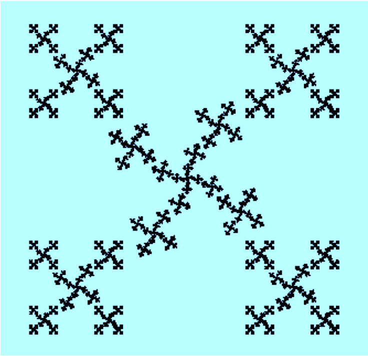
$$f_1(x,y) = (\tfrac{1}{2}x, \tfrac{1}{2}y);\ f_2(x,y) = (\tfrac{1}{2}x + \tfrac{1}{2}, \tfrac{1}{2}y);\ f_3(x,y) = (\tfrac{1}{2}x, \tfrac{1}{2}y + \tfrac{1}{2}).$$

$$E = f_1(E) \cup f_2(E) \cup f_3(E)$$



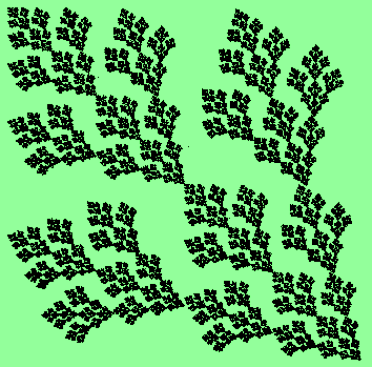


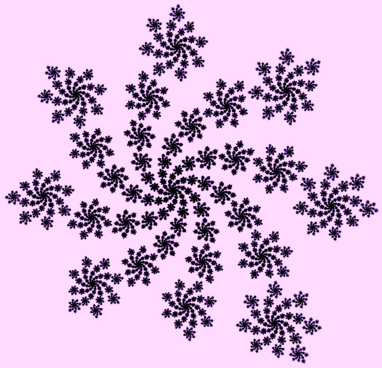
finite rotations



dense rotations

Self-similar sets





More self-similar sets

Dimension of self-similar sets

Let E be a self-similar set as above satisfying

$$E = \bigcup_{i=1}^{m} f_i(E). \qquad (*)$$

Provided the union in $(*)$ is disjoint or the open set condition holds,

$$\dim_H E = s \quad \text{where} \quad \sum_{i=1}^{m} r_i^s = 1,$$

where r_i is the similarity ratio of f_i.

E.g. Hausdorff dimension of the Sierpiński triangle is given by $3(1/2)^s = 1$ or $s = \log 3/\log 2$.

Let E be the 1-dimensional Sierpínski triangle, so $\dim_H E = 1$.

For projections onto the line with slope θ:

(a) if $\theta = p/q$ is rational,

and $p + q \not\equiv 0 \,(\mathrm{mod}\ 3)$

$\dim_H \mathrm{proj}_\theta E < 1$;

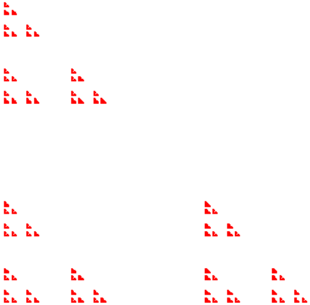
and $p + q \equiv 0 \,(\mathrm{mod}\ 3)$

$\mathrm{proj}_\theta E$ contains an interval,

(b) if θ is irrational,

$\dim_H \mathrm{proj}_\theta E = 1$ but $\mathcal{L}(\mathrm{proj}_\theta E) = 0$.

(Kenyon 1997, Hochman 2014)



Self-similar sets with finite rotations

Theorem (Farkas 2014) Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with finite rotations and with $\dim_H E < 1$. Then there is at least one value of θ such that $\dim_H \text{proj}_\theta E < \dim_H E$

Self-similar sets with dense rotations

Theorem (Peres & Shmerkin 2009, Hochman & Shmerkin 2012)
Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Then

$$\dim_H \text{proj}_\theta E = \min\{\dim_H E, 1\} \text{ for } \textbf{all } \theta.$$

Self-similar sets with dense rotations

Theorem (Peres & Shmerkin 2009, Hochman & Shmerkin 2012)
Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Then

$$\dim_H \mathrm{proj}_\theta E = \min\{\dim_H E, 1\} \text{ for } \textbf{all } \theta.$$

Requires ideas from ergodic scenery flows, CP chains, r-scale entropy, Marstrand's theorem, ...

Theorem (Peres & Shmerkin 2009, Hochman & Shmerkin 2012)
Let $E \subset \mathbb{R}^2$ be a self-similar set defined by a family $\{f_1, \ldots, f_m\}$ of similarities with dense rotations. Then

$$\dim_H \mathrm{proj}_\theta E = \min\{\dim_H E, 1\} \text{ for } \textbf{all } \theta.$$

Requires ideas from ergodic scenery flows, CP chains, r-scale entropy, Marstrand's theorem, ...

Corollary (Hochman & Shmerkin 2012) With E as above, for **all** non-singular C^1 functions $h : N \to \mathbb{R}$, where N is a neighbourhood of E,

$$\dim_H h(E) = \min\{\dim_H E, 1\}.$$

This follows using the result for projections locally, noting that at at very fine scales h 'looks like' a projection of a small copy of E in some direction.

Measure of projections

These methods are unable to show that for a self-similar set E with $\dim_H E > 1$ all projections have positive measure. However, this is very nearly so in the plane.

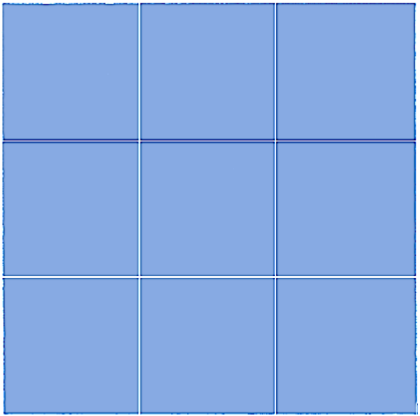
Measure of projections

These methods are unable to show that for a self-similar set E with $\dim_H E > 1$ all projections have positive measure. However, this is very nearly so in the plane.

Theorem (Shmerkin & Solomyak 2014) Let $E \subset \mathbb{R}^2$ be the self-similar attractor of an IFS with dense rotations with $1 < \dim_H E < 2$. Then $\mathcal{L}(\text{proj}_\theta E) > 0$ for all θ except (perhaps) for a set of θ of Hausdorff dimension 0.

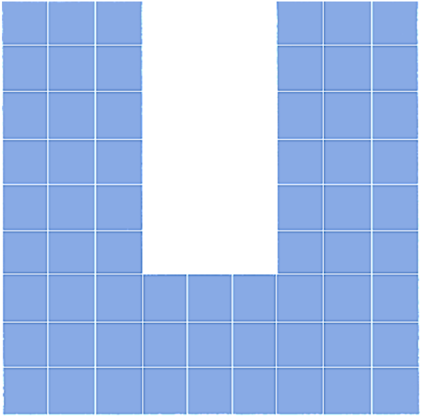
The proof involves a careful analysis of how the Fourier transform of the projections of a natural measure on E varies with θ.

Mandelbrot percolation on a square



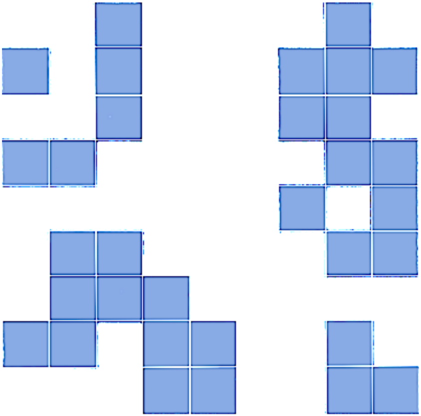
- Squares are repeatedly divided into 3×3 subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

Mandelbrot percolation on a square



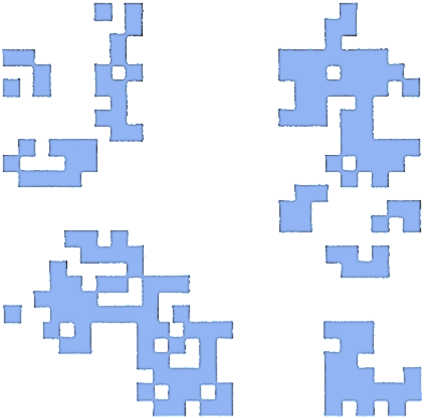
- Squares are repeatedly divided into 3×3 subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

Mandelbrot percolation on a square



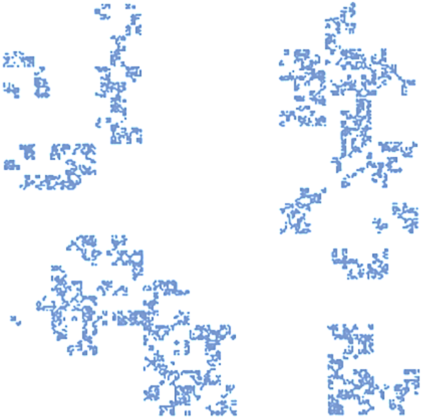
- Squares are repeatedly divided into 3×3 subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

Mandelbrot percolation on a square



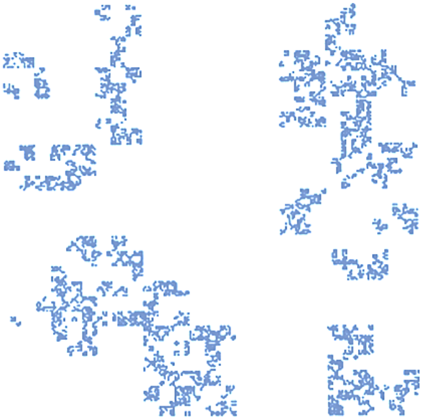
- Squares are repeatedly divided into 3×3 subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

Mandelbrot percolation on a square



- Squares are repeatedly divided into 3×3 subsquares
- Each square is retained independently with probability p ($\simeq 0.6$).

Mandelbrot percolation on a square



If $p > 1/M^2$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_H E_p = 2 + \log p / \log M$ almost surely.

Projections of Mandelbrot percolation

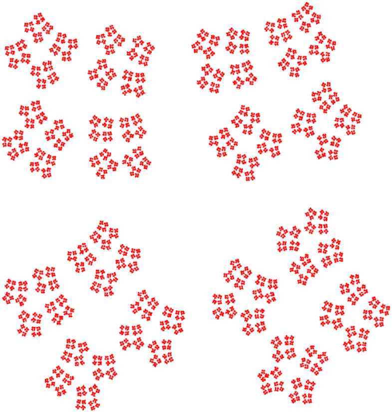
Theorem (Rams & Simon, 2012) Let E_p be the Mandelbrot percolation set obtained by dividing squares into $M \times M$ subsquares, each square being retained with probability $p > 1/M^2$. Conditional on $E_p \neq \emptyset$:

(i) $\dim_H \text{proj}_\theta E_p = \min\{\dim_H E_p, 1\}$ for **all** $\theta \in [0, \pi)$,

(ii) if $p > 1/M$ then $\dim_H E_p > 1$, and, for **all** $\theta \in [0, \pi)$, $\text{proj}_\theta E_p$ contains an interval and in particular $\mathcal{L}(\text{proj}_\theta E_p) > 0$.

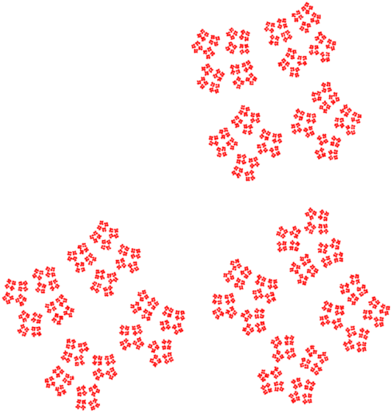
Proof depends on a geometrical analysis of how lines intersect the grid squares.

Percolation on a self-similar set



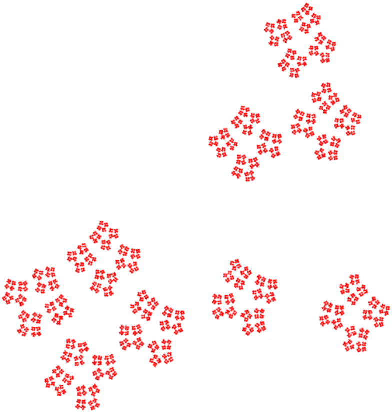
Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



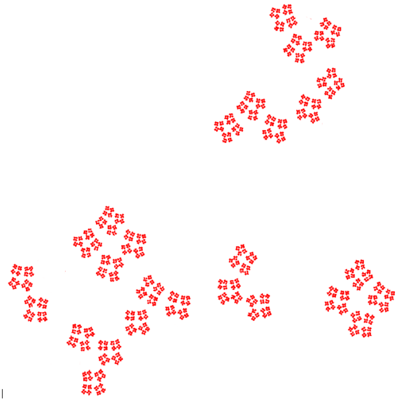
Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



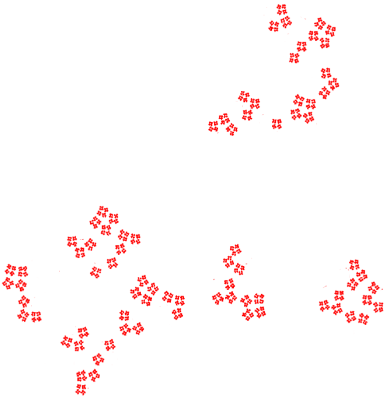
Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



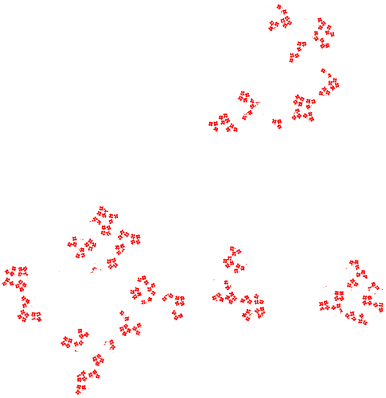
Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



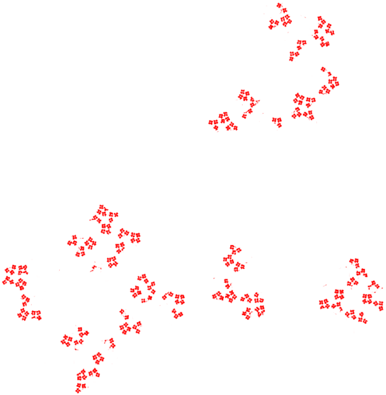
Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



Start with a self similar set. At each stage of the iterated construction, retain each component with probability p.

Percolation on a self-similar set



If $p > 1/m$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_H E_p = s$, where $p \sum_{i=1}^{m} r_i^s = 1$

Projection of percolation sets

If $p > 1/m$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_H E_p = s$, where $p\sum_{i=1}^m r_i^s = 1$, with r_i the scaling component of f_i.

Theorem (Jin & F, 2014) Let E have dense rotations and let $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely

$$\dim_H \text{proj}_\theta E_p = \min\{\dim_H E_p, 1\} \text{ for } \textbf{all } \theta.$$

Projection of percolation sets

If $p > 1/m$ then $E_p \neq \emptyset$ with positive probability, conditional on which $\dim_H E_p = s$, where $p\sum_{i=1}^m r_i^s = 1$, with r_i the scaling component of f_i.

Theorem (Jin & F, 2014) Let E have dense rotations and let $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely

$$\dim_H \text{proj}_\theta E_p = \min\{\dim_H E_p, 1\} \text{ for } \textbf{all } \theta.$$

This is a special case of a more general result on random multiplicative cascades on self-similar sets.

Let

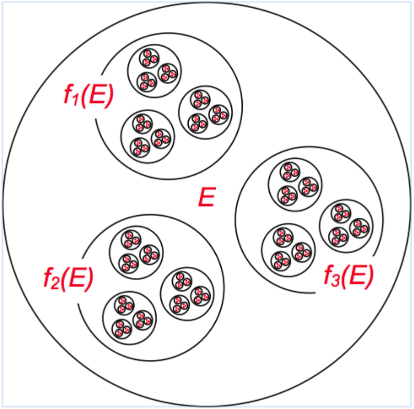
$$W = (W_1, \ldots, W_m) \in [0, \infty)^m$$

be a random vector such that $\sum_{i=1}^m \mathbb{E}(W_i) = 1$. For each $k \geq 0$ and $(i_1, \ldots, i_k) \in \{1, \ldots, m\}^k$ let

$$W = (W_1^{i_1,\ldots,i_k}, \ldots, W_m^{i_1,\ldots,i_k}) \in [0, \infty)^m$$

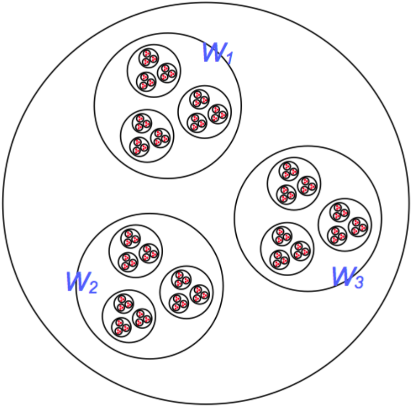
be i.i.d copies of W.

Random multiplicative cascades on self-similar sets



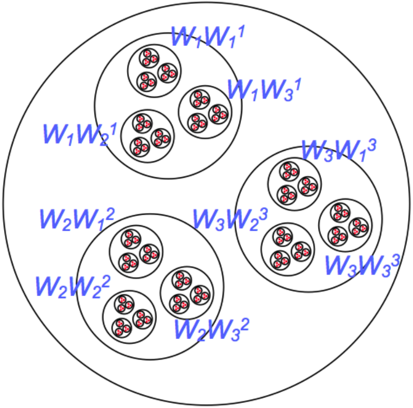
Iterative construction of a self-similar set $E = \bigcup_{i=1}^{3} f_i(E)$

Random multiplicative cascades on self-similar sets



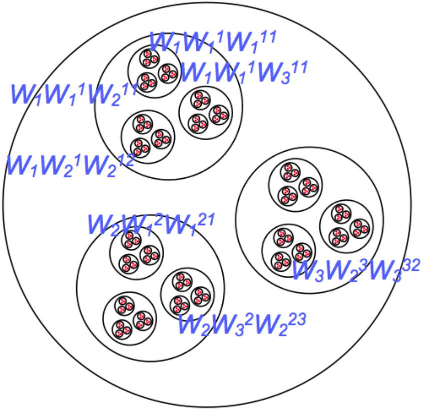
Construction of a random cascade measure on E

Random multiplicative cascades on self-similar sets



Construction of a random cascade measure on E

Random multiplicative cascades on self-similar sets



Construction of a random cascade measure on E

Random multiplicative cascades on self-similar sets

The condition $\sum_{i=1}^m \mathbb{E}(W_i) = 1$ means that for each $i_1, \ldots, i_j$ the sequence of measures $(\mu_k)_{k\geq j}$ given by

$$\mu_k(f_{i_1} \circ \cdots \circ f_{i_j}(E)) \equiv W_{i_1} W_{i_2}^{i_1} W_{i_3}^{i_1,i_2} \cdots W_{i_j}^{i_1,\ldots,i_{j-1}} \sum_{i_{j+1},\ldots,i_k} W_{i_{j+1}}^{i_1,\ldots,i_j} \cdots W_{i_k}^{i_1,\ldots,i_{k-1}}$$

is a martingale, so a.s. μ_k converges on basic sets to an additive set function which extends to the random cascade measure μ on E, where μ is non-trivial with positive probability provided that $\sum_{i=1}^m \mathbb{E}(W_i^p) < \infty$ for some $p > 1$.

Special cases:

Mandelbrot multiplicative cascades
Natural measures on fractal percolation sets
Branching constructions

Projections of random multiplicative cascades

Theorem (F & Jin 2014) Let μ be a random multiplicative cascade on a self-similar set $E \subset \mathbb{R}^2$ with dense rotations. Almost surely, conditional on $\mu \neq 0$,

$$\dim_H \mathrm{proj}_\theta \mu = \min\{\dim_H \mu, 1\} \text{ for } \textbf{all } \theta.$$

Projections of random multiplicative cascades

Theorem (F & Jin 2014) Let μ be a random multiplicative cascade on a self-similar set $E \subset \mathbb{R}^2$ with dense rotations. Almost surely, conditional on $\mu \neq 0$,

$$\dim_H \text{proj}_\theta \mu = \min\{\dim_H \mu, 1\} \text{ for } \textbf{all } \theta.$$

Very brief idea of proof First show almost surely:

(i) μ is exact dimensional,

(ii) for almost all θ, $\text{proj}_\theta \mu$ is exact dimensional with

$$\dim_H \text{proj}_\theta \mu = \min\{\dim_H \mu, 1\}.$$

This uses an ergodic-theoretic argument to show that the natural 'shift-like' operator T on

$$\Omega := \left\{ (i_i, i_2, \ldots), \left(W^{\mathbf{i}} : \mathbf{i} \in \bigcup_k \{1, \ldots, m\}^k \right) \right\}$$

is invariant and ergodic with respect to the Peyrière measure on Ω.

Then the space and operator are extended to include a rotation element which is ergodic by the compact group extension theorem.

Projections of random multiplicative cascades

Fix $0 < \rho < 1$. For each $q \in \mathbb{N}$, the IFS

$$\{f_{i_1} f_{i_2} \cdots f_{i_k} : r_{i_1} r_{i_2} \cdots r_{i_{k-1}} > \rho^q \geq r_{i_1} r_{i_2} \cdots r_{i_k}\}$$

defines the same attractor E, and redefining the random vector W appropriately we can get the same distribution of measures μ with respect to this refined IFS. With H_r denoting 'r-scale entropy', for the corresponding map T_{ρ^q}, almost surely

$$\begin{aligned} \dim_H \mathrm{proj}_\theta \mu &\geq \frac{\mathbb{E}(H_{\rho^q}(\mathrm{proj}_\theta \mu))}{q \log(1/\rho) - c} - O(1/q) \quad \text{for all } \theta \\ &\to \dim_H \mathrm{proj}_\theta \mu \quad \text{as } q \to \infty \\ &= \min\{\dim_H \mu, 1\} \quad \text{for almost all } \theta. \end{aligned}$$

Now use lower-semicontinuity.

Projections of random multiplicative cascades

Theorem (F & Jin 2014) Let μ be a random multiplicative cascade on a self-similar set $E \subset \mathbb{R}^2$ with dense rotations. Almost surely, conditional on $\mu \neq 0$,

$$\dim_H \mathrm{proj}_\theta \mu = \min\{\dim_H \mu, 1\} \text{ for } \textbf{all } \theta.$$

Projections of random multiplicative cascades

Theorem (F & Jin 2014) Let μ be a random multiplicative cascade on a self-similar set $E \subset \mathbb{R}^2$ with dense rotations. Almost surely, conditional on $\mu \neq 0$,

$$\dim_H \mathrm{proj}_\theta \mu = \min\{\dim_H \mu, 1\} \text{ for **all** } \theta.$$

Corollary Let E have dense rotations and let E_p be percolation on E where $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely

$$\dim_H \mathrm{proj}_\theta E_p = \min\{\dim_H E_p, 1\} \text{ for **all** } \theta.$$

[Take $W = (W_1, \ldots, W_m) = (r_1^s X_1, \ldots, r_m^s X_m)$,
where $X_1, \ldots, X_p$ are i.i.d. with $X_i = 1$ (prob p), $= 0$ (prob $1 - p$), and check that $\dim_H E_p = \dim_H \mu$.]

Projections of random multiplicative cascades

Theorem (F & Jin 2014) Let μ be a random multiplicative cascade on a self-similar set $E \subset \mathbb{R}^2$ with dense rotations. Almost surely, conditional on $\mu \neq 0$,

$$\dim_H \text{proj}_\theta \mu = \min\{\dim_H \mu, 1\} \text{ for } \textbf{all } \theta.$$

Corollary Let E have dense rotations and let E_p be percolation on E where $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely

$$\dim_H \text{proj}_\theta E_p = \min\{\dim_H E_p, 1\} \text{ for } \textbf{all } \theta.$$

[Take $\quad W = (W_1, \ldots, W_m) = (r_1^s X_1, \ldots, r_m^s X_m)$,
where $X_1, \ldots, X_p$ are i.i.d. with $X_i = 1$ (prob p), $= 0$ (prob $1 - p$), and check that $\dim_H E_p = \dim_H \mu$.]

Corollary Let E have dense rotations and let $p > 1/m$. Then, conditional on $E_p \neq \emptyset$, almost surely:

For **all** non-singular C_1 functions $h : N \to \mathbb{R}$, where N is a neighbourhood of E,

$$\dim_H h(E_p) = \min\{\dim_H E_p, 1\}.$$

Projection of percolation sets

As in the case of deterministic self-similar sets of dimension > 1, for percolation on self-similar sets we cannot quite guarantee projections of positive length in all directions.

Theorem (F & Jin 2015) Let $E_p \subset \mathbb{R}^2$ be percolation on a self-similar set E with dense rotations with $1 < \dim_H E_p < 2$. Then, almost surely, $\mathcal{L}(\text{proj}_\theta E_p) > 0$ for all θ except for a set of θ of Hausdorff dimension 0.

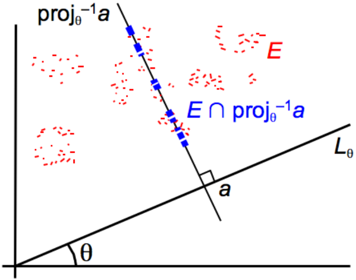
Projection of percolation sets

As in the case of deterministic self-similar sets of dimension > 1, for percolation on self-similar sets we cannot quite guarantee projections of positive length in all directions.

Theorem (F & Jin 2015) Let $E_p \subset \mathbb{R}^2$ be percolation on a self-similar set E with dense rotations with $1 < \dim_H E_p < 2$. Then, almost surely, $\mathcal{L}(\text{proj}_\theta E_p) > 0$ for all θ except for a set of θ of Hausdorff dimension 0.

The proof considers projections of a natural random measure μ on E_p. We express $\widehat{\mu} = \widehat{\mu^0}\ \widehat{\mu^1}$ in such a way that almost surely $\dim_H \text{proj}_\theta \mu^0 = 1$ for all θ, and for all θ except for a set of dimension 0, $|\widehat{\text{proj}_\theta \mu^1}(t)| \leq c|t|^{-\epsilon}$, which implies that $\text{proj}_\theta \mu$ is absolutely continuous.

Marstrand's section theorem



Theorem (Marstrand 1954) Let $E \subset \mathbb{R}^2$ be a Borel set with $\dim_H E > 1$. Then,

(i) for all $\theta \in [0, \pi)$

$$\dim_H(E \cap \mathrm{proj}_\theta^{-1} a) \leq \dim_H E - 1 \text{ for } \mathcal{L}\text{-almost all } a,$$

(ii) for $\mathcal{L}$-almost all $\theta \in [0, \pi)$

$$\mathcal{L}\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) \geq \dim_H E - 1\} > 0.$$

Marstrand's section theorem

Question: When is

$$\mathcal{L}\left\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) \geq \dim_H E - 1\right\} > 0$$

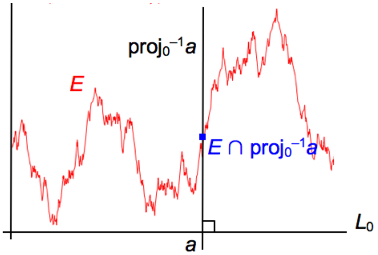
for 'all θ' rather than 'almost all θ' ?

Marstrand's section theorem

Question: When is

$$\mathcal{L}\big\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) \geq \dim_H E - 1\big\} > 0$$

for 'all θ' rather than 'almost all θ' ?



The graph of a function can have dimension as large as 2. However, $E \cap \mathrm{proj}_0^{-1} a$ is a single point for all $a \in L_0$, so $\dim_H(E \cap \mathrm{proj}_0^{-1} a) = 0$.

Marstrand's section theorem

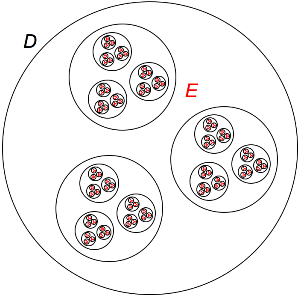
Question: When is

$$\mathcal{L}\big\{a \in L_\theta : \dim_H(E \cap \text{proj}_\theta^{-1} a) \geq \dim_H E - 1\big\} > 0$$

for 'all θ' rather than 'almost all θ' ?

Furstenberg (2008) addressed this for self-similar sets with finite rotations. He introduced the notion of dimension conservation for when this was the case.

For the case of dense rotations, to obtain such results on sections of a deterministic self-similar E we use the results on projections of random percolation subsets E_p.



To find the Hausdorff dimension of subsets of a self-similar set E it is enough to take covers by 'basic sets' of the iterative construction of E, that is sets of the form $U_{\mathbf{i}} = f_{i_1} \circ \cdots \circ f_{i_k}(D)$.
Thus, for $F \subset E$:

$$\dim_H F = \inf \Big\{ s : \text{for all } \epsilon > 0 \text{ there are basic sets } \{U_i\} \text{ with } F \subset \bigcup_i U_i \text{ and } \sum_i (\operatorname{diam} U_i)^s < \epsilon \Big\}.$$

We can use (random) percolation sets to test the dimension of deterministic (i.e. non-random) subsets of self-similar sets.

Lemma Let E be a self-similar set constructed iteratively with basic sets $\{U_{\mathbf{i}}\}$. Let E_p be obtained by fractal percolation on the basic sets $\{U_{\mathbf{i}}\}$, and suppose for some $\alpha > 0$

$$\mathbb{P}\{U_{\mathbf{i}} \text{ survives the percolation process}\} \leq c(\text{diam } U_{\mathbf{i}})^{\alpha} \text{ for all } \mathbf{i}.$$

If $F \subset E$ and $\dim_H F < \alpha$ then $E_p \cap F = \emptyset$ almost surely.

– In particular, if $F \subset E$ and $E_p \cap F \neq \emptyset$ with positive probability, then $\dim_H F \geq \alpha$.

Lemma Let E_p be obtained from the self-similar set E by fractal percolation on the basic sets $\{U_{\mathbf{i}}\}$, and suppose

$$\mathbb{P}\{U_{\mathbf{i}} \text{ survives the percolation process}\} \leq c(\operatorname{diam} U_{\mathbf{i}})^\alpha \text{ for all } \mathbf{i}.$$

If $F \subset E$ and $\dim_H F < \alpha$ then $E_p \cap F = \emptyset$ almost surely.

Proof Given $\epsilon > 0$ let $\mathcal{I}$ be a family of indices such that

$$F \subset \bigcup_{\mathbf{i}\in\mathcal{I}} U_{\mathbf{i}} \quad \text{and} \quad \sum_{\mathbf{i}\in\mathcal{I}} (\operatorname{diam} U_{\mathbf{i}})^\alpha < \epsilon.$$

Then

$$\mathbb{E}\big(\#\{\mathbf{i}\in\mathcal{I} : E_p \cap U_{\mathbf{i}} \neq \emptyset\}\big) \leq \sum_{\mathbf{i}\in\mathcal{I}} \mathbb{P}\{U_{\mathbf{i}} \text{ survives}\} \leq c\sum_{\mathbf{i}\in\mathcal{I}} (\operatorname{diam} U_{\mathbf{i}})^\alpha < c\epsilon,$$

so

$$\mathbb{P}\big(E_p \cap F \neq \emptyset\big) \leq \mathbb{P}\Big(E_p \cap \bigcup_{\mathbf{i}\in\mathcal{I}} U_{\mathbf{i}} \neq \emptyset\Big) < c\epsilon.$$

This is true for all $\epsilon > 0$, so $\mathbb{P}\big(E_p \cap F \neq \emptyset\big) = 0$.

Corollary Let E be a self-similar set constructed iteratively using a hierarchy of basic sets $\{U_{\mathbf{i}}\}$. Let E_p be the random set obtained by some fractal percolation process on the $\{U_{\mathbf{i}}\}$ and suppose that for some $\alpha > 0$

$$\mathbb{P}\{U_{\mathbf{i}} \text{ survives the percolation process}\} \leq c(\text{diam}\, U_{\mathbf{i}})^{\alpha} \text{ for all } \mathbf{i}.$$

For each θ, if

$$\mathbb{P}\Big\{\mathcal{L}(\text{proj}_\theta E_p) > 0\Big\} > 0,$$

then

$$\mathcal{L}\Big\{a \in L_\theta : \dim_H(E \cap \text{proj}_\theta^{-1} a) \geq \alpha\Big\} > 0.$$

Proof Let
$$S = \{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) < \alpha\}.$$

For each $a \in S$, taking $F = E \cap \mathrm{proj}_\theta^{-1} a$ in the lemma,

$$E_p \cap \mathrm{proj}_\theta^{-1} a = E_p \cap E \cap \mathrm{proj}_\theta^{-1} a = \emptyset$$

almost surely. In other words, for each $a \in S$, $a \notin \mathrm{proj}_\theta E_p$ with probability 1.

By Fubini's theorem, with probability 1, $a \notin \mathrm{proj}_\theta E_p$ for $\mathcal{L}$-almost all $a \in S$.

Hence, with positive probability,

$$0 < \mathcal{L}(\mathrm{proj}_\theta E_p) = \mathcal{L}\big((\mathrm{proj}_\theta E_p) \setminus S\big) \leq \mathcal{L}\big((\mathrm{proj}_\theta E) \setminus S\big).$$

Sections of self-similar sets

Theorem (F & Jin 2015) Let $E \subset \mathbb{R}^2$ be a self-similar set with dense rotations with $1 < \dim_H E \leq 2$. Then, for all $\epsilon > 0$:

$$\mathcal{L}\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) > \dim_H E - 1 - \epsilon\} > 0$$

for all θ except for a set of θ of Hausdorff dimension 0.

Sections of self-similar sets

Theorem (F & Jin 2015) Let $E \subset \mathbb{R}^2$ be a self-similar set with dense rotations with $1 < \dim_H E \leq 2$. Then, for all $\epsilon > 0$:

$$\mathcal{L}\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) > \dim_H E - 1 - \epsilon\} > 0$$

for all θ except for a set of θ of Hausdorff dimension 0.

Proof for E a self-similar set made up of m copies at scale r:
Take $p = m^{-1} r^{-1-\epsilon}$ so $\dim_H E_p > 1$ with positive probability, in which case by the projection result, $\mathcal{L}(\mathrm{proj}_\theta E_p) > 0$ for all θ except for a set of Hausdorff dimension 0. Also

$$\mathbb{P}\{U_\mathbf{i} \text{ survives the percolation process}\} \leq c(\mathrm{diam}\, U_\mathbf{i})^{\log p/\log r}$$

so by the Corollary

$$\mathcal{L}\{a \in L_\theta : \dim_H(E \cap \mathrm{proj}_\theta^{-1} a) \geq \alpha\} > 0$$

where
$\alpha = \log p/\log r = (-\log m - (1+\epsilon)\log r)/\log r = \dim_H E - 1 - \epsilon.$

Sections of self-similar sets

Theorem (F & Jin 2015) Let $E \subset \mathbb{R}^2$ be a self-similar set with dense rotations with $1 < \dim_H E \leq 2$. Suppose E is connected or $\text{proj}_\theta E$ is an interval for all θ. Then for all $\epsilon > 0$

$$\dim_H \left\{ a \in L_\theta : \dim_B (E \cap \text{proj}_\theta^{-1} a) > \dim_H E - 1 - \epsilon \right\} = 1$$

for **all** θ.

Theorem (F & Jin 2015) Let $E \subset \mathbb{R}^2$ be a self-similar set with dense rotations with $1 < \dim_H E \leq 2$. Suppose E is connected or $\mathrm{proj}_\theta E$ is an interval for all θ. Then for all $\epsilon > 0$

$$\dim_H \left\{ a \in L_\theta : \dim_B(E \cap \mathrm{proj}_\theta^{-1} a) > \dim_H E - 1 - \epsilon \right\} = 1$$

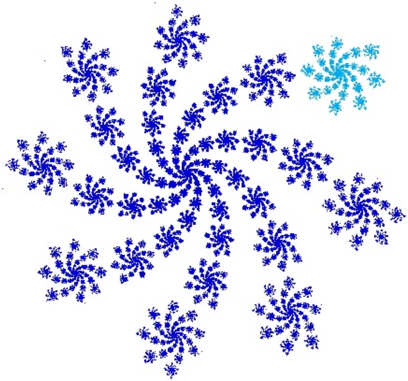
for **all** θ.

Theorem (F & Jin 2015) Let E_p be the Mandelbrot percolation set obtained by dividing squares into $M \times M$ subsquares, each square being retained with probability $p > 1/M^2$. Then, for all $\epsilon > 0$, conditional on $E_p \neq \emptyset$,

$$\mathcal{L}\left\{ a \in L_\theta : \dim_H(E_p \cap \mathrm{proj}_\theta^{-1} a) > \dim_H E_p - 1 - \epsilon \right\} > 0$$

for **all** θ.

Proof. Similar idea, using that the intersection of two independent percolation sets $E_p \cap E_q$ has the same distribution as the single percolation set E_{pq}.



Thank you!